Homework 1, Honors Calculus II
due 9/20/18

Please hand in your home work before class, have it neatly written, organized (the grader will not decipher your notes), stapled, with your name and student ID on top.

Problem 1. Assuming that we know that the area of a circle of radius R is $R^2\pi$ (which can be shown by elementary geometry), calculate an approximation of π by using the function

$$f\colon [0,1] \to \mathbb{R}\,, \quad f(x) = \sqrt{1-x^2}$$

whose graph is a quarter-circle of radius $R = 1$. Use equidistant partition into five subintervals and choose as your evaluation points the *left* end points in each subinterval (in class we used the right end points of the subintervals), and calculate the Riemann sum for that partition.

Problem 2. Here two ways to calculate the sum of the squares $\sum_{k=1}^{n} k^2$ of the first n integers.

(i) One way(out of many) to find a formula for the sum of the squares $\sum_{k=1}^{n} k^2$ of the first n integers is the following: interprete k^2 as the volume of a slab of hight one with square base of side length k. Then we see that our sum is the volume of the sum of all those slabs, that is, it presents a volume. That means, that our sum aught to be some cubic polynomial in n, that is, we can conjecture

$$\sum_{k=1}^{n} k^2 = an^3 + bn^2 + cn + d$$

Now d has to be zero (why?), and then there are the three unknowns a, b, c to be determined. Choosing three values for n for which the sum can be easily calculated, this gives three linear equations for the three unknowns a, b, c. If these equations can be solved, then you have a found a formula.

(ii) Use the binomial formula $(k+1)^3 = \ldots$ to deduce $(k+1)^3 - k^3 = \ldots$ (fill in the correct expressions for the dots) and then sum this last identity over $k = 1, \ldots, n$.

(iii) **extra points** Use the previous idea to calculate the sum of cubes $\sum_{k=1}^{n} k^3$. Can you see a method to calculate $\sum_{k=1}^{n} k^4$ and etc.?

Problem 3. Find the area under the cubic $y = x^3$ over the interval $[0,1]$. You will need to find a formula for the sum of the cubes $\sum_{k=1}^{n} k^3$ of the first n numbers to do that.

Problem 4. Here a function (necessarily not continuous) for which the integral does not exist. Let $f\colon [0,1] \to \mathbb{R}$ be given by $f(x) = 1$ if x is a rational number, and $f(x) = 0$ if x is an irrational number (try to graph this function....). Show that the Riemann sum for f does not converge (that is, its limit depends on the choice of partition) by choosing a sequence of partitions and evaluation points so that the Riemann sum converges to zero, and then choose another sequence of partitions and evaluation points so that the Riemann sum converges to one.

Problem 5. Show that the infinite sum

$$\sum_{k=1}^{\infty} \frac{1}{k+n} := \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k+n} = \log 2\,.$$

We always use logarithms to base e unless specifically indicated. Two hints: think of a Riemann sum for which function? What is the area under the hyperbola $y = 1/x$ over the interval $[1, 2]$?

Problem 6. Find an anti-derivative of $f(x) = \log x$ and calculate the area under the graph $y = \log x$ over the interval $[1, 2]$.

Problem 7. Calculate the indefinite integrals (that is, find anti-derivatives)

$$\int \sin^n(x)dx = ? \quad \text{and} \quad \int \cos^n(x)dx = ?$$