HOMEWORK 2, M 331.2
DUE 9/26/16

Please hand in your home work before class, have it neatly written, organized (the grader will not decipher your notes), stapled, with your name and student ID on top.

Problem 1. A sky diver of mass 90 kg (including equipment) jumps from an airplane at an altitude of 3000 meters. Assume that we model the air drag proportional to velocity. The air drag coefficient γ before opening the chute is $\gamma = 20$ and after opening the chute $\gamma = 200$.

(i) Write down the ODE modeling the parachuter.
(ii) How long (in seconds) does it take the parachuter (without opening the chute) to be within 1% of terminal velocity v_1?
(iii) Assuming that the skydiver wants to open his parachute at 300 meters above ground, how long will he be able to dive without his parachute open?
(iv) Making the (reasonable) approximation that at 300 meters above ground the skydiver has the terminal velocity v_1 calculated above, and then opens the chute (we assume this happens instantaneously), how long does it take to come within 1% of terminal velocity v_2 with the chute open?
(v) How long is the skydiver in the air with the parachute open?
(vi) How long is the skydiver in the air from the start of his jump to the landing?
(vii) What are the terminal velocities v_1 and v_2 (in meters per second).
(viii) Plot carefully (graph paper, correct scale, labels etc) the graph of the velocity function $v(t)$ and the position function $y(t)$ during the total sky diving event.

Problem 2. On 19 September 1991 an extraordinary archaeological discovery was made at a high-altitude mountain pass (Tisenjoch, 3210 m) of the Ötztal Alps near the Austrian-Italian border. Two mountain hikers, after having scaled the Finail Peak (3516 m) that day were on their way back to the Similaun mountain hut (3019 m) located at the lowest part of a mountain ridge connecting the Finail Peak with the Similaun (3607). This ridge forms the border between Austria (to the north) and Italy (to the south). As the hikers approached a shallow ice-filled depression along the ridge, they were startled by seeing the body of a man sticking half-way out from the ice. Unusual climatic conditions in the summer of 1991 (including dust from Sahara resulting in enhanced melting of snow) had partly freed the body from his icy grave. The Iceman was later nicknamed "Ötzi", after the mountain range where he was found. Calculate when the Iceman died by using carbon dating: measurements showed that the corpse contained 53% of the amount of the radioactive isotope C^{14} found in living organisms. The half life of C^{14} is 5730 years.

Problem 3. The population of the US was 8.6 Million in 1820 and 40 Million in 1897.

(i) Write down the differential equation for population growth according to the following model: the rate of change of the population at any given time is proportional to the population present at that time.

(ii) Calculate the number of people in the US according to your model in the year 2003. How well does this growth model work: the actual population of the US in 2003 was 291 Million.
(iii) How long does it take for the population in the US to double according to your growth model?

Problem 4. One (rather simple) model for the spreading of an infectious disease is the following: in a population of N individuals the rate of change of infected individuals $P'(t)$ at a given time t is proportional (with some constant $k > 0$, the *spread rate*, depending on the type of disease) to the number of possible contacts of infected and non-infected individuals, that is, to the number of infected individuals $P(t)$ times the number of uninfected individuals $N - P(t)$ at time t.

(i) Write down the differential equation for this model.
(ii) Interpret the equilibrium solutions in words.
(iii) Find the general solution of your ODE (in terms of N and k and a constant of integration C) and draw a graph of the shape of the general solution.
(iv) At which point in time t_{max} does the disease spread the fastest? What property characterizes t_{max}? Can you find a formula for t_{max} in terms of N, k and C?
(v) Assume that in a population of $N = 1000$ individuals there are initially 20 infected individuals and after 10 days there are 150. Calculate the spread rate k of this disease.
(vi) How long would it take until 60% of the population is infected? 90% affected?
(vii) At which point in time is this disease spreading fastest?

Problem 5. Consider the ODE

$$y' = (y-1)(y+1)$$

(i) Draw a slope line picture in the (t, y)-plane and draw the graph of the "driving term" (the right hand side $(y-1)(y+1)$ of the ODE).
(ii) Determine all the equilibrium solutions and characterize them as stable, unstable or semi-stable.
(iii) Determine all the solutions $y(t)$ of the ODE and graph some of them in the (t, y)-plane.
(iv) Find the solutions $y_1(t)$, $y_2(t)$ and $y_3(t)$ with initial conditions $y_1(0) = -1$, $y_2(0) = 0$ and $y_3(0) = 2$ respectively.

Use graphing paper with reasonably correct scales for your pictures.

Problem 6. Consider the ODE $y' = y^2$.

(i) What are the equilibrium solutions? Are they stable, unstable or semi-stable?
(ii) Find all solutions $y(t)$ of the ODE.
(iii) Calculate the solution with initial condition $y(0) = 1$ and determine the time s when the solution "blows up", i.e., goes to infinity, i.e., when $\lim_{t\to s} y(t) = \pm\infty$.

Problem 7. Solve the ODE $y' = \frac{y^2}{t^2-1}$ with initial condition $y(0) = 1$.

Problem 8. Consider the ODE $(y')^2 = y^2 - 1$.

(i) Draw a slope line picture in the (t, y)-plane.

(ii) What are the equilibrium solutions? Are they stable, unstable or semi-stable?

(iii) Find all solutions $y(t)$ of the ODE and graph some of them in the (t, y)-plane.

(iv) Calculate the solution with initial condition $y(0) = 2$. Can you find solutions to any initial condition $y(0) = c$ where c is an arbitrary number?

Use graphing paper with reasonably correct scales for your pictures.

Problem 9. Find all solutions to the ODE $y' = 2te^{y-t^2}$. Which of those solutions satsify $y(1) = 1$?