Homework 2, Differential Geometry
due 2/10/17

Please hand in your home work before class, have it neatly written, organized (the grader will not decipher your notes), stapled, with your name and student ID on top.

Problem 1. Calculate the curvature function κ of an ellipse. How many maxima and minima does κ have? Where do they occur?

Problem 2. Calculate the curvature κ of one branch of a hyperbola. How many maxima/minima does κ have and where do they occur?

Problem 3. Draw the curve $\gamma(t) = (\cos(t), \sin(2t))$ for $t \in [0, 2\pi]$. Calculate the curvature κ of this curve and determine its maxima/minima. Does γ have inflection points, i.e. points where $\kappa(t_0) = 0$?. If so, where are they?

Problem 4. Let $\gamma\colon I \to \mathbb{R}^n$ be a regular smooth curve. Show that the map γ is locally injective, that is for all $t_0 \in I$ there is some $\epsilon > 0$ so that γ is injective when restricted to $(t_0 - \epsilon, t_0 + \epsilon) \cap I$. Give an example of a regular curve which is not (globally) injective.

Problem 5. Let $\gamma\colon I \to \mathbb{R}^2$ be a smooth regular planar curve and assume $0 \in I$. Take $t \neq 0$ in I such that also $-t \in I$ and consider the unique circle $C(t)$ (which could also be a line) containing the 3 points $\gamma(0), \gamma(-t), \gamma(t)$. Show that the curvature of $C(t)$ converges to the curvature $\kappa(0)$ of γ at $t = 0$ when $t \to 0$. This also shows that the curvature is a geometric quantity, i.e. parametrization invariant.

Problem 6. Let $\gamma\colon I \to \mathbb{R}^2$ be a regular (smooth) curve and $\tilde{\gamma} = \gamma \circ \varphi$ with $\varphi\colon \tilde{I} \to I$ a reparametrization, Show, by using the general formula $\kappa = \det(\gamma', \gamma'')/||\gamma'||^3$ for the curvature of a regular curve, that $\tilde{\kappa} = \pm\kappa \circ \phi$, where $\pm$ depends on whether φ is orientation preserving (+) or reversing (-).

Problem 7. Show that the length of a curve γ in $\mathbb{R}^n$ is invariant under Euclidean motions, i.e. show that $L(A\gamma) = L(\gamma)$ for $Ax = Rx + a$ an Euclidean motion of $\mathbb{R}^n$.

Problem 8. A velociraptor is spotting you and goes after you. There is a shelter in the direction perpendicular to the line between you and the raptor when he spots you. So you run in the direction of the shelter at a constant speed v. The raptor is pursuing you (also at a constant speed w) adjusting at each moment in time t his direction to the line between you and him (pursue in the direction of the line of sight). Calculate the curve $\gamma(t)$ the raptor is following. Do some calculations (taking realistic speeds into account) whether you would make it to your shelter (play with the distance to the shelter compared to your and the raptor's running speeds). Draw a picture of the situation.