Homework 4, Honors Calculus II
due 10/4/18

Please hand in your home work before class, have it neatly written, organized (the grader will not decipher your notes), stapled, with your name and student ID on top. All integrals have to be computed without using symbolic calculators. You may use a calculator only to verify a result, and for numerical calculations which you cannot do on paper or in your head.

Problem 1. The hyperbolic cos and sin functions are defined as

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \quad \text{and} \quad \sinh(x) = \frac{e^x - e^{-x}}{2}.$$

Verify the following identities:

(i) $\cosh^2(x) - \sinh^2(x) = 1$.
(ii) $\cosh^2(x) + \sinh^2(x) = \cosh(2x)$.
(iii) $\cosh'(x) = \sinh(x)$ and $\sinh'(x) = \cosh(x)$.

Problem 2. Use the graph parametrization $\gamma(t) = (t, \sqrt{R^2 - t^2})$ of the upper semi-circle of radius $R > 0$ to compute the length of the semi-circle.

Problem 3. Find a curve $\gamma\colon I \to \mathbb{R}^2$, $\gamma(t) = (\gamma_1(t), \gamma_2(t))$, which traces out *one whole branch* of the hyperbola $x^2 - y^2 = 1$. *Hint*: look at the example of the circle from class—not from the previous problem— and put on your hyperbolic glasses...

Problem 4. Do the following experiment: take your wooden cutting board, hammer in two nails at some distance $D > 0$, take a string of length $L > D$ and tie each end of the string to one of the nails (make the string long enough so that the string is not taut after you tied it to the nails, I would take L to be at least $2D$), then take a pencil and use it to push the sting taut, and keeping it taught move the pencil over the cutting board in a circular motion. You will get an oval like curve on your cutting board. Here some questions for you to answer:

(i) The oval like curve you get is an ellipse $(\frac{x}{a})^2 + (\frac{y}{b})^2 = 1$, provided you align the x-axis along the line connecting the two nails and let the origin be the midpoint between the two nails. Here a and b are determined by D and L by some formulas you have to determine. Draw a picture.
(ii) The location of the nails are called the *focal points* F_1, F_2 of the ellipse. Imagine a light ray emanating from one of the focal points, say F_1. Verify that if the light ray reflects off the ellipse (by the usual reflection law, namely incoming angle equals outgoing angle measured with respect to the tangent line at the point of the ellipse where the light ray hits) and ends up in the other focal point F_2. Draw a picture depicting this scenario.
(iii) Now move the focal point F_2 further and further away, i.e. make D larger and larger (then also $L > D$ has to become larger and larger and you will need to have more real estate on your cutting board, or start using the wood floor of your room, or go to some sport field etc.). What happens to the light rays emanating from F_1 when D gets very large? Draw pictures for the various larger and larger D's.
(iv) Can you provide a computational argument that the ellipse becomes a parabola when $D \to \infty$? Combine this with your previous observation about what the light rays emanating from F_1 do in this case, to deduce

that incoming light rays parallel to the axis of a parabola, when reflected off the parabola, all meet at the same point F_1 on the axis of the parabola? The principle of the parabolic mirror.

(v) If you didn't succeed to prove (iv), do this explicit calculation: consider the parabola $y = x^2$ and prove that any vertical line $x = a$ (other than the y-axis, i.e. $a \neq 0$) reflects off the parabola and intersects the y-axis at a fixed pint $P = (0, f)$ (what will f have to be?) independent of what $a \neq 0$ is.

Problem 5. As a sanity check of our definition of length for curves, check that a straight line segment connecting two *arbitrary* points $P \neq Q$ in the plane has indeed length $||P - Q||$. To do this, find a curve γ parametrizing the line segment $\overline{PQ}$, and use the definition of length as an integral of speed.

Problem 6. Consider the logarithmic spiral $\gamma(t) = e^t(\cos t, \sin t)$ for $t \in \mathbb{R}$. Draw a reasonably accurate picture.

(i) Calculate the velocity, speed, and acceleration of γ for any value of $t \in \mathbb{R}$.
(ii) What are the velocity, speed, and acceleration of γ at $t = 0$?
(iii) Calculate the length $L(\gamma)$ of the spiral over the time interval $[a, b]$.
(iv) What happens to the length when $a \to -\infty$?

Problem 7. Given a smooth function $f\colon [a, b] \to \mathbb{R}$, we defined the graph parametrization to be the curve $\gamma(t) = (t, f(t))$ for $t \in [a, b]$. Find the formula for the length of γ in terms of f. Apply this formula to calculate the length of the parabola $y = x^2$ over the interval $[a, b]$.

Problem 8. We are told that the acceleration of a curve γ is constant, say $g = (g_1, g_2) \in \mathbb{R}^2$. Find all curves γ in the plane which have this acceleration, that is, solve the equation $\gamma''(t) = g$. If you had done this around Newton's time, your name would be in all the text books. What is the physical relevance of this problem?