Homework 4, Advanced Calculus
due 2/22/17

Please hand in your home work before class, have it neatly written, organized (the grader will not decipher your notes), stapled, with your name and student ID on top.

Problem 1. Consider the following function: $f(x_1, x_2) = 1$ for $x_2 = x_1^2 \neq 0$ and $f(x_1, x_2) = 0$ at all other points in the plane.

(i) Draw the graph of this function.
(ii) Show that the function is not continuous at $x = (0, 0)$.
(iii) Show that all directional derivatives at $x = (0, 0)$ in any direction $v \in \mathbb{R}^2$ exist and calculate them.
(iv) Is the function differentiable at $x = 0$? Give a reason for your answer.

Problem 2. Consider the function

$$f(x_1, x_2) = \begin{cases} \frac{x_1 x_2}{\sqrt{x_1^2 + x_2^2}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

defined on $\mathbb{R}^2$.

(i) Calculate the Jacobi matrix $Df(x) = (\frac{\partial f}{\partial x_1}(x), \frac{\partial f}{\partial x_2}(x))$ at any point $x \in \mathbb{R}^2$.
(ii) Verify that the partial derivatives of f are not continuous at $x = 0$.

Problem 3. For each of the functions below use the theorem about differentiability rules (and knowledge of differentiable functions in 1 variable from Calculus I) to provide an argument why these functions are differentiable everywhere. Calculate the derivative $Df(x)$ in each case. Use the formula (valid for differenatiable functions) $Df(x; v) = Df(x)v$ to calculate the directional derivative at the given point $\mathring{x}$ in the direction of v:

(i) $f(x_1, x_2) = (x_1 - x_2)^2$, $\mathring{x} = (1, 1)$, $v = (2, 3)$
(ii) $f(x_1, x_2) = x_1^3 - x_2$, $\mathring{x} = (1, 0)$, $v = (1, 1)$
(iii) $f(x_1, x_2, x_3) = x_1^2 + x_2^2$, $\mathring{x} = (1, 1, 2)$, $v = (2, 3, 4)$
(iv) $f(x_1, x_2) = x_1 x_2$, $\mathring{x} = (0, 1)$, $v = (1, 0)$
(v) $f(x_1, x_2, x_3) = x_1^2 - x_2^2 - x_3$, $\mathring{x} = (1, 0, 1)$, $v = (0, 1, 2)$

Problem 4. Consider the function

$$f(x_1, x_2) = \begin{cases} \frac{x_1^3 x_2}{x_1^4 + x_2^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

on the domain $\mathbb{R}^2$.

(i) At which points $x \in \mathbb{R}^2$ is the function differentiable. Provide reasoning for your answer (quote results, properties and theorems from class).
(ii) At points $x \in \mathbb{R}^2$ where f is differentiable, calculate a general formula (some expression in x and v) for the directional derivative $Df(x; v)$ in the direction of $v = (v_1, v_2)$ by using the formula $Df(x; v) = Df(x)v$.

Problem 5. In the following items calculate $D(f \circ g)$ in two ways: first, using the chain rule, and second, by calculating $h = f \circ g$ explicitly and then calculating Dh:

(i) $f(x_1, x_2) = x_1^2 - x_2^2$ and $g(t) = (\cosh t, \sinh t)$
(ii) $f(x_1, x_2) = x_1 + x_2$ and $g(t_1, t_2, t_3) = (e^{t_2 t_3}, t_1^2 + t_2^2)$

(iii) $f(x_1, x_2) = x_1^3$ and $g(t_1, t_2) = (t_1, t_2^3)$
(iv) $f(x_1, x_2) = \cos(x_1^2 + x_2^2)$ and $g(t) = (\cos(t), \sin(2t))$
(v) $f(x_1, x_2) = \frac{1}{x_1 - x_2}$ for $x_1 \neq x_2$ and $g(t_1, t_2, t_3, t_4) = (1 + t_1^2 + t_3^2, -2 - t_2^2)$

Problem 6. Calculate $D(f \circ g)$ for $f(x) = ||x||^2$, $x \in \mathbb{R}^n$ and $g(t) = (t_1, t_2^2, t_3^3, \ldots, t_n^n)$, $t \in \mathbb{R}^n$.

Problem 7. Consider the unit radius 2-sphere $S^2 = \{x \in \mathbb{R}^3 \,;\, ||x|| = 1\}$.

(i) Show that the map

$$g(\alpha, \beta) = (\cos\alpha\cos\beta, \sin\alpha\cos\beta, \sin\beta)$$

has its image in S^2, i.e., for all values of (α, β) the value $g(\alpha, \beta) \in S^2$.

(ii) Draw the lines $\alpha = c$ and $\beta = \tilde{c}$ under the map g (i.e., the curves $g(c, \beta)$ and $g(\alpha, \tilde{c})$ on S^2 when the angles α and β vary) for various constants $c, \tilde{c}$ (e.g. $c = 0, \pi/2, \pi, 3\pi/2$ etc. and $\tilde{c} = -\pi/2, -\pi/4, 0, \pi/4, \pi/2$ etc.) and interpret those by imagining S^2 to be our earth.

(iii) Show that g is differentiable and calculate $Dg(\alpha, \beta)$.

(iv) Assume a temperature distribution over the earth's surface given by

$$T(x_1, x_2, x_3) = 80 - 120x_3^2$$

Calculate the rate of change of T per unit angle along the curves $g(c, \beta)$ and $g(\alpha, \tilde{c})$.