HW 4 Solutions

Problem 1 : $y' = -ky + A + B\cos(\omega t)$

homogeneous solution : $y_H(t) = C \cdot e^{-kt}$

particular solution: variation of constants works always but leads to tedious integration by parts. Better use "undetermined coefficients method" since inhomogeneity is given by simple function

$$A + B\cos(\omega t)$$

i.e., sum of a degree 0 polynomial At^0 & a trig function. So our Ansatz for $y_P(t)$ should be replicating this function:

$$y_P(t) = a + \alpha\cos(\omega t) + \beta\sin(\omega t)$$

($\boxed{\text{Note}}$: ~~even~~ though inhomogeneity has only a $\cos(\omega t)$, the Ansatz requires a sum of $\cos(\omega t)$ & $\sin(\omega t)$)

$$y_P' = -\alpha\omega\sin(\omega t) + \beta\omega\cos(\omega t)$$

insert into our ODE, equate left & right sides to determine the constants a, α, β:

$$\overbrace{-\alpha\omega\sin(\omega t) + \beta\omega\cos(\omega t)}^{y_P'} \overset{!}{=} -k\overbrace{(a + \alpha\cos\omega t + \beta\sin\omega t)}^{y_P} + A + B\cos\omega t$$

compare t^0-terms: $0 = -ka + A \rightsquigarrow a = \frac{A}{k}$

compare $\sin(\omega t)$-terms: $-\alpha\omega = -k\beta$

$\cos(\omega t)$-terms: $\beta\omega = -k\alpha + B$

solve for α & β:

$$\alpha = \frac{kB}{k^2+\omega^2}, \quad \beta = \frac{B\omega}{k^2+\omega^2}$$

Thus, we obtain

$$y_P(t) = \frac{A}{k} + \frac{B}{k^2+\omega^2}\left[k\cos\omega t + \omega\sin\omega t\right]$$

and

$$y(t) = y_P(t) + y_H(t) = \frac{A}{k} + \frac{B}{k^2+\omega^2}[\cdots] + Ce^{-kt}$$

rewrite the general solution

$$y(t) = \frac{A}{k} + \frac{B}{k^2+\omega^2}\left[k\cos\omega t + \omega \sin\omega t\right] + Ce^{-kt}$$

Initial condition $y(0)=0$ gives

$$\frac{A}{k} + \frac{Bk}{k^2+\omega^2} + C = 0 \rightsquigarrow \boxed{C = -\frac{(A+B)k^2 + A\omega^2}{k(k^2+\omega^2)}}$$

Let's calculate the numbers: $k=0.1$, $A=B=1$, $\omega=\pi/3$

$$\frac{A}{k} = 10, \quad \frac{B}{k^2+\omega^2} = 0.9, \quad C = -\frac{2\cdot 10^{-2} + (\pi/3)^2}{10^{-1}\left(10^{-2}+(\pi/3)^2\right)} = \frac{1.12}{0.11}$$

$$\boxed{C = -10.18}$$

For $t \to \infty$ (or large enough), the homogeneous solution dies off rather quickly, so we only see the particular oscillatory solution.

The externally administered substance crests when $\cos(\omega t)$ has a maximum, which is

$$\omega t = 2n\pi, \quad n \in \mathbb{N} \text{ natural \#}$$

$$\frac{\pi}{3} t = 2n\pi \quad \text{or} \quad t = 6n, \; n \in \mathbb{N}$$

Thus, once $y_H(t)$ has died off sufficiently, we see $y(t)$ oscillating like

$$\frac{B}{k^2+\omega^2}(k\cos\omega t + \omega \sin\omega t)$$

~~To~~ and we have to see where this oscillation has its crest. There are a number of ways of doing this:

- Use a formula to write

  $$k\cos\omega t + \omega\sin\omega t = D\cdot\cos(\omega t + \delta)$$

  to get the phase shift δ. We will soon derive such a formula.

- If you don't know such a formula, use Calc I and find maxima of

$$k\cos\omega t + \omega\sin\omega t$$

by setting its derivative to zero:

$$-k\omega\sin\omega t + \omega^2\cos\omega t = 0$$

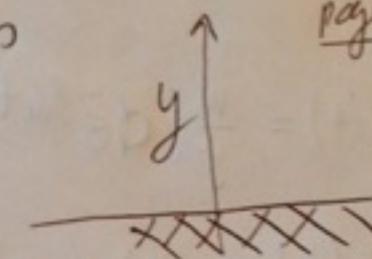
$$\tan(\omega t) = \frac{\omega}{k} = 10.47$$

$$\omega t = \arctan(10.47) = 1.47 \; (+n\pi)$$

Now you would have to make sure this is the crest and if so, then you would know that the solution $y_p(t)$ (the "response" after waiting long enough) crests about 1½ hours later than the externally administered agent.

End Problem 1

HW 4 Solutions



Problem 2: $v = y'$,

$$mv' = -mg - \gamma v$$

1) $\gamma = 0$: $v' = -g \rightsquigarrow v(t) = -gt + v_0$, $v_0 = 0$

$$y(t) = \int v(t)\,dt = -\frac{g}{2}t^2 + y_0, \quad y_0 = 100\text{ m}, \quad g = 9.81\,\frac{\text{m}}{\text{s}^2}$$

Thus

$$y(t) = -\frac{9.81}{2}t^2 + 100$$

Find t so that $y(t) = 0$:

$$0 = -\frac{9.81}{2}t^2 + 100$$

$$t^* = \sqrt{\frac{200}{9.81}} = 4.5 \text{ seconds.}$$

2) $\gamma = 1$: $v' = -g - \frac{\gamma}{m}v$

$$\int \frac{dv}{g + \frac{\gamma}{m}v} = -\int dt + C$$

$$v(t) = \frac{m}{\gamma}\left(c\, e^{-\gamma/m\, t} - g\right), \quad v(0) = 0 \rightsquigarrow c = g$$

$$v(t) = \frac{mg}{\gamma}\left(e^{-\gamma/m\, t} - 1\right)$$

Since $y' = v$, we get

$$y(t) = \int v(t)\,dt = \frac{mg}{\gamma}\left[-\frac{m}{\gamma} e^{-\gamma/m\, t} - t\right] + C$$

$$y(0) = 100 \rightsquigarrow C = \frac{m^2 g}{\gamma^2} + 100$$

Set $y(t) = 0$ to find t:

$$\frac{mg}{\gamma}\left[-\frac{m}{\gamma} e^{-\gamma/m\, t} - t + 100 + \frac{m^2}{\gamma^2} g\right] = 0$$

Solving (via calculator or zeros finder etc.)

get $t = 4.6$ seconds.

— End problem 2 —

Problem 3 $\quad Q' = kQ(A-Q) = -kQ(Q-A)$

$$\int \frac{dQ}{Q(Q-A)} = -\int k\,dt = -kt + C$$

$$\overset{\ast}{=}\ -\frac{1}{A}\int\left(\frac{1}{Q} - \frac{1}{Q-A}\right)dQ \quad \text{thus}$$

$$\int\left(\frac{1}{Q} - \frac{1}{Q-A}\right)dQ = Akt + C$$

$$\ln\left|\frac{Q}{Q-A}\right| = Akt + C \quad (\ast)$$

Now $Q(0) = 1$, so $\frac{Q(0)}{Q(0)-A} \approx \frac{1}{1-10^6} \cong -10^{-6}$

$$\ln|-10^6| = 6 \cdot \ln 10 = \underbrace{10^6}_{A} \cdot \underbrace{10^{-2}}_{k} + C$$

$$C = -9986$$

This allows us to calculate the time it takes to learn 75% by using $(\ast)$

$$\ln \frac{3/4 \cdot 10^6}{10^6 - 3/4\, 10^6} = \underbrace{10^6 \cdot 10^{-2}}_{10^4} t - 9986$$

$$\ln \frac{3/4\, 10^6}{1/4\, 10^6} = \underbrace{\ln (3 \times 10^6)}_{14.9}$$

$$\underbrace{9986 + 14.9}_{\simeq 10^4} = 10^4 t$$

Thus $t \sim 1$.

If time is in years, then it would take us about 1 year.

Equilibria: $Q \equiv 0$: you never will know anything

$Q \equiv A = 10^6$: you forever will know every thing

general solution: exponentiate $*$:

$$\frac{Q}{Q-A} = \tilde{C} e^{Akt} \rightsquigarrow Q(t) = \frac{A \tilde{C} e^{Akt}}{\tilde{C} e^{Akt} - 1}$$

with $\tilde{C} = e^C$.

When do we learn fastest?

This happens when Q' ~~grows~~ has a maximum, i.e. when $Q'' = 0$ (assuming, and defining, that Q' has only maxima etc ...).

Since we have a formula for Q', it is easy to calculate Q'':

$$Q'' = \frac{d}{dt} Q' = \frac{d}{dt} kQ(A-Q) =$$

$$= kQ'A - 2kQQ' = \quad \text{insert formula for } Q' = kQ(A-Q)$$

$$= Ak^2 Q(A-Q) - 2k^2 QQ(A-Q) =$$

$$= k^2 Q(A-Q)[A - 2Q]$$

Thus one learns fastest when $Q = \frac{A}{2}$, that is at the halfway point. Like with the 75% question, we could use $*$ to calculate the time to reach 50% ....:

$$\ln \frac{\frac{1}{2}10^6}{\frac{1}{2}10^6} = 10^6 \cdot 10^{-2} t - 9986$$

$$\ln 1 = 0 \qquad t = \frac{9986}{10^4}$$

$t = 0.99$ years

a bit less than ~~…~~

leaving 75%

I think due to the steep learning curve (k is far too big I guess) the solution curve $Q(t)$ is almost vertically up, thus no big difference between leaving 50% & 75%.

Important to take away: for logistic model the growth (or decline) is fastest at the half way point.

—·—· End problem 3 —·—

HW 4 Solutions

Problem 4

$$y' - y = \underbrace{1 + t^2 + \cos(t)}_{f(t) \text{ the inhomogeneity}}$$

(i) $y_H(t) = \tilde{C}_1 e^t$

(ii) $y_p(t) = ?$ Again, could do variations of constants etc but run into integration by parts. Since $f(t)$ is ~~is~~ composed of simple functions

$$\underbrace{1+t^2}_{\substack{\text{polynomial} \\ \text{of degree 2}}} + \underbrace{\cos(t)}_{\text{trig function}}$$

we are led to the Ansatz:

$$y_p(t) = \underbrace{a + bt + ct^2}_{\substack{\text{quadratic} \\ \text{polynomial}}} + \underbrace{A\cos t + B\sin t}_{\substack{\text{combination of} \\ \sin \text{ \& } \cos}}$$

We now calculate y_p' and insert everything into our ODE to calculate the unknown coefficients by comparing like powers of t & sin & cos terms:

(Remark: ~~[illegible]~~ for easier organization some people prefer to do the polynomial & trig parts seperately):

$$y_p' = b + 2ct - A\sin t + B\cos t$$

Insert into ODE:

$$\underbrace{b+2ct-A\sin t+B\cos t}_{y_p'} - (\underbrace{a+bt+ct^2+A\cos t+B\sin t}_{y_p}) \overset{!}{=} 1+t^2+\cos t$$

comparing:

$$\left.\begin{array}{ll} t^0\text{-terms}: & b-a=1 \\ t\text{-terms}: & 2c-b=0 \\ t^2\text{-terms}: & -c=1 \end{array}\right\} \begin{array}{l} c=-1 \\ b=-2 \\ a=-3 \end{array}$$

$$\left.\begin{array}{ll} \cos t\text{-terms}: & B-A=1 \\ \sin t\text{-terms}: & -A-B=0 \end{array}\right\} \left.\begin{array}{l} B=-A \\ -2A=1 \end{array}\right\} \begin{array}{l} A=-\tfrac{1}{2} \\ B=\tfrac{1}{2} \end{array}$$

Thus

$$\boxed{y_p(t) = -3-2t-t^2-\tfrac{1}{2}\cos t+\tfrac{1}{2}\sin t}$$

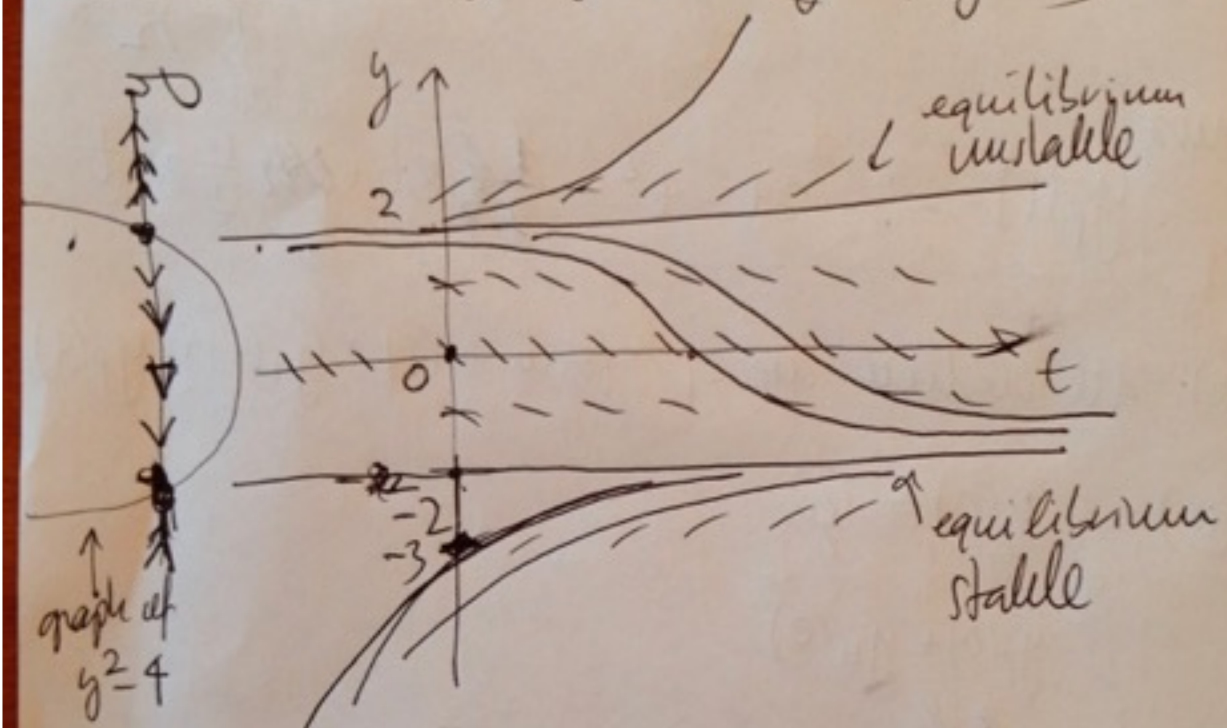
(iii) All solutions are of form $y(t) = y_H(t) + y_p(t)$

(iv) $y(0) = 0$

$$y(0) = y_p(0) + y_H(0) = \left(-3-\tfrac{1}{2}\right) + \tilde{c} = 0 \Rightarrow \tilde{c} = \tfrac{7}{2}$$

— End of problem 4. —

Problem 5. $y' = y^2 - 4 = (y-2)(y+2)$



$$\frac{dy}{(y-2)(y+2)} = dt$$

partial fractions again: $\frac{1}{(y-2)(y+2)} = \frac{1}{4}\left(\frac{1}{y-2} - \frac{1}{y+2}\right)$

$$\int \frac{dy}{(y-2)(y+2)} = \frac{1}{4}\left(\ln(y-2) - \ln(y+2)\right)$$
$$= \frac{1}{4}\ln\frac{y-2}{y+2} = t + c$$

$$\ln\frac{y-2}{y+2} = 4t + C$$

$$\frac{y-2}{y+2} = Ce^{4t}$$

$y(0) = -3$: $\frac{-3-2}{-3+2} = C \leadsto C = 5$

explicitly solve for y:

$$y - 2 = (y+2)Ce^{4t}$$

$$y(1 - Ce^{4t}) = 2Ce^{4t} + 2 = 2(Ce^{4t} + 1)$$

$$\boxed{y(t) = 2\cdot\frac{5\cdot e^{4t} + 1}{1 - 5\cdot e^{4t}}}$$

— · — End problem 5 — · —

Problem 6:

$$\frac{dy}{dt} = \underbrace{2y^2 + t^2y^2}_{y^2(2+t^2)} \quad , y(0) = -\frac{1}{2}$$

$$\underbrace{\int \frac{dy}{y^2}}_{= -\frac{1}{y}} = \int (2+t^2)\,dt = 2t + \frac{1}{3}t^3 + C$$

$$y = -\frac{1}{C + 2t + \frac{1}{3}t^3} = -\frac{1}{2+2t+\frac{1}{3}t^3}$$

$$y(0) = -\frac{1}{2} \rightsquigarrow C = 2$$

Since $2+2t+\frac{1}{3}t^3$ is cubic polynomial, it has (by intermediate value theorem) at least one real zero $\rightsquigarrow$ $y(t)$ develops a vertical asymptote at $t = -0.88462$

$t = -0.88462$

(one can check other two zeros are not real)

Problem 7: $y'' - 2y' - 3y = 0$

$$\lambda^2 - 2\lambda - 3 = 0$$

$$\lambda_{1,2} = 1 \pm \sqrt{1-(-3)} = 1 \pm \sqrt{4} = 1 \pm 2 = \begin{cases} 3 \\ -1 \end{cases}$$

general solution: $y(t) = c_1e^{3t} + c_2e^{-t}$

initial conditions:

$$1 = y(0) = c_1 + c_2$$

$$y' = 3c_1e^{3t} - c_2e^{-t}$$

$$-1 = y'(0) = 3c_1 - c_2$$

Thus

$$\left.\begin{aligned} c_1 + c_2 &= 1 \\ 3c_1 - c_2 &= -1 \end{aligned}\right\} \quad \left\{\begin{aligned} c_1 &= 1 - c_2 \\ 3(1-c_2) - c_2 &= -1 \end{aligned}\right\}$$

$$4 = 4c_2 \rightsquigarrow c_2 = 1 \rightsquigarrow c_1 = 0$$

$$\boxed{y(t) = e^{-t}}$$

Problem 8:
$$y'' + y' - 2y = 0$$

$$\lambda^2 + \lambda - 2 = 0$$

$$\lambda_{1,2} = -\frac{1}{2} \pm \sqrt{1/4 + 2} = -\frac{1}{2} \pm \sqrt{9/4}$$

$$\lambda_{1,2} = \begin{cases} 1 \\ -2 \end{cases}$$

$$\boxed{y(t) = c_1 e^{t} + c_2 e^{-2t}}$$

— END —