Homework 6, Differential Geometry
due 3/10/17

Please hand in your home work before class, have it neatly written, organized (the grader will not decipher your notes), stapled, with your name and student ID on top.

Problem 1. Let $\gamma_s\colon I \to \mathbb{R}^n$, $s \in (-\epsilon, \epsilon)$, $\epsilon > 0$, be a variation with compact support $K \subset \mathring{I}$ of a *regular* curve $\gamma = \gamma_0$. Show that there exists some $0 < \delta \leq \epsilon$ such that γ_s is a regular curve for all $s \in (-\delta, \delta)$. Thus, we may assume w.l.o.g. that any variation of a regular curve consists of regular curves.
Hint: go over this carefully and try to understand each step: for any $t \in I$, since γ is regular, $||\gamma'(t)|| \geq M(t) > 0$ for some positive number $M(t)$ possibly depending on t. By continuity of γ'_s in s we thus know that there exists some $\delta(t) > 0$ such that $||\gamma'_s(t)|| \geq \frac{1}{2}M(t) > 0$ for $s \in (-\delta(t), \delta(t))$ (a continuous function cannot drop to zero immediately, try to prove this). The idea now is to take the minimum of all those $\delta(t)$ as t ranges over I, and hope this minimum is > 0. Unless some compactness is used, this won't work (e.g. $I = \mathbb{R}$ and $\delta(t) \sim 1/t$). You will have to use the fact that by the compact support of the variation γ_s you only have to show regularity of γ_s for $t \in K$ (why?). The defining property of a compact set is, that *any* covering by open intervals (or more generally, open sets) has a finite subcover, that is, already finitely many of those covering intervals will cover K....

Problem 2. Show that the bending energy

$$W(\gamma) = \int_I \kappa^2(t)||\gamma'(t)||dt$$

on regular planar curves $\gamma\colon I \to \mathbb{R}^2$ is parametrization invariant, i.e. $W(\gamma \circ \varphi) = W(\gamma)$ for any reparametrization $\varphi\colon \tilde{I} \to I$.

Problem 3. Consider the kinetic energy functional

$$E(\gamma) = \frac{1}{2}\int_I ||\gamma'(t)||^2 dt$$

for a curve $\gamma\colon I \to \mathbb{R}^n$.

(i) Show that E is not parametrization invariant, i.e. $E(\gamma \circ \varphi) \neq E(\gamma)$ in general.
(ii) Show that γ is a critical point of E under compactly supported variations γ_s if and only if $\gamma'' = 0$, i.e., $\gamma(t) = x_0 + tv$ is a constant speed straight line (notice, no assumptions on the regularity of the curve is needed).
(iii) What are the minima of E?

In the last step of (ii), when carrying out the calculus of variations, you will need the following, more general, version of the "Fundamental Lemma of Variational Calculus": let $f\colon I \to \mathbb{R}^n$ be continuous then

$$\int_I < f(t), g(t) > dt = 0$$

for all continuous $g\colon I \to \mathbb{R}^n$ compactly supported in $\mathring{I}$, if and only if $f \equiv 0$. Prove this statement.