Homework 7, Honors Calculus II
10/25/2018

This home work discusses some of the basics of the "partial fraction decomposition". The intent is not to derive the most general formula (which can be looked up), but to understand the principle behind it.

The problem to be solved is the following: how can we integrate a rational function $f(x) = \frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials. Recall that a polynomial is a function of the form $P(x) = a_n x^n + \cdots + a_1 x + a_0$ with $a_k \in \mathbb{R}$ real constants, the coefficients of the polynomial $P(x)$. Provided that the leading coefficient $a_n \neq 0$ (which we always assume when we write down a polynomial), the *degree* of the polynomial $\deg P = n$. So a degree 1 (or linear) polynomial is of the form $P(x) = ax+b$, a degree 2 (or quadratic) polynomial is of the form $P(x) = ax^2 + bx + c$ and so on. Important to keep in mind is, that a degree n polynomial has $(n+1)$-many coefficients! Thus, to determine a degree n polynomial, one needs to prescribe $(n+1)$-many points its graph goes through, which will lead to $(n+1)$-many linear equations for the coefficients a_k, which generally can be solved.

Problem 1. Find the quadratic polynomial whose graph contains the points $(0,0)$, $(1,1)$ and $(-1,4)$. Notice that there is *no* linear polynomial containing those three points (after all, a line is determined by two points). And there will be many cubic (degree 3) polynomials containing those three points (find at least two different such cubic polynomials). The reason is, that a cubic polynomial has 4 coefficients, and the given 3 points only give 3 linear equations, not enough to pin down all 4 coefficients.

Let's continue with our problem of integrating rational functions $\frac{P(x)}{Q(x)}$. If $\deg P \geq \deg Q$ (numerator degree greater or equal than denominator degree), we perform long division and arrive at

$$\frac{P(x)}{Q(x)} = R(x) + \frac{\tilde{P}(x)}{Q(x)}$$

with $R(x)$ a polynomial of degree $\deg R = \deg P - \deg Q \geq 0$ and a remainder term $\frac{\tilde{P}(x)}{Q(x)}$ where now $\deg \tilde{P} < \deg Q$. Since we can integrate a polynomial, this means we only have to understand how to integrate rational functions whose numerator degree is strictly smaller than its denominator degree.

Problem 2. Carry out the long division for the rational functions

$$f(x) = \frac{3x^4 + 2x^2 - 5x + 1}{x^2 + x + 1} \quad \text{and} \quad g(x) = \frac{3x^4 + 2x^2 - 5x + 1}{x^4 + 1}$$

and identify the polynomial $R(x)$ and $\tilde{P}(x)$ in both cases and check if the degrees are as stated above.

Next we have to remind ourselves of the *Fundamental Theorem of Algebra* (whose proof, ironically, uses Analysis and Topology; there is no known proof using only Algebra, and perhaps there never will be). This theorem states that any polynomial with real coefficients can be written as

$$P(x) = a(x - x_1)^{n_1}(x - x_2)^{n_2} \cdots (x - x_r)^{n_r}(x^2 + b_1 x + c_1)^{m_1} \cdots (x^2 + b_l x + c_l)^{m_l}$$

where $a, b_i, c_i \in \mathbb{R}$ are some real constants, $x_1, \ldots x_r$ are distinct real numbers, the zeros of the polynomial $P(x)$, and $n_1, \ldots, n_r$ are positive integers, the corresponding multiplicities of the zeros. The quadratic factors $x^2 + b_1x + c_1$ and so on have *no* real zeros, and $m_1, \dots, m_l$ are positive integers. This is quite ugly and if we had complex numbers at our disposal, all of this would boil down to the statement that a degree n polynomial has exactly n-many zeros (counted with multiplicities) over the complex numbers. Those quadratic terms in our formula cannot be further written as products of linear factors, since they have no real zeros. Using this knowledge, we get the relation $n = n_1 + \cdots + n_r + 2m_1 + \cdots + 2m_l$.

Problem 3. Find the above decomposition for the following polynomials (basic idea here is to guess a zero and etc.) and compare to the general formula, identify the constants a, b_i, c_i, the integers n_i and m_i, and the zeros x_i:

(i) $P(x) = 4x^4 - 3x^2 - 1$
(ii) $P(x) = x^3 + 1$
(iii) $P(x) = x^4 - 1$

Let's use this to get a feel how we can integrate rational functions $\frac{P(x)}{Q(x)}$ with $\deg P < \deg Q$: we start with the simple case when there are none of the quadratic terms in the denominator, that is, when the denominator polynomial $Q(x) = (x - x_1)^{n_1}(x-x_2)^{n_2} \cdots (x-x_r)^{n_r}$ factors into linear terms over the reals, in other words, when all its zeros are real, possibly repeating. Then we have $\deg Q = n_1 + \cdots + n_r > \deg P$ and we make the following Ansatz:

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x - x_1} + \frac{A_2}{(x - x_1)^2} + \cdots + \frac{A_{n_1}}{(x - x_1)^{n_1}} + \text{same for the other linear factors in } Q(x)$$

Multiplying by $Q(x)$ yields and identity

$$P(x) = R(x)$$

where $R(x)$ is a polynomial of degree at most $\deg Q - 1$ (check this, why?), and since $\deg P < \deg Q$, both sides are polynomials of degree at most $\deg Q - 1$. Since such a polynomial is determined by $\deg Q$ many conditions, and we have exactly that many unknowns $A_1, A_2 ....$, we can solve for those A_i.

Problem 4. Integrate the following rational functions:

(i) $f(x) = \frac{x^3}{(x-1)(x+1)^3}$
(ii) $f(x) = \frac{x}{(x+2)^2(x+1)^2}$
(iii) $f(x) = \frac{1}{x^2(x+5)(x-2)}$
(iv) $f(x) = \frac{x^3+2x+1}{(x+1)^2}$
(v) $f(x) = \frac{x^4+1}{x+1}$