HOMEWORK 8, M 331.2
DUE 11/9/16

Please hand in your home work before class, have it neatly written, organized (the grader will not decipher your notes), stapled, with your name and student ID on top.

Problem 1 (Motion of a pendulum with small elongation)**.** The angle $y(t)$, measured in radiants from the vertical in counter clockwise direction, of a pendulum suspended on a string (rod etc.) of length L (in meters) with pendant of some mass m satisfies the ODE

$$y'' + \frac{g}{L}y = 0$$

where g is the gravitational constant (notice, that the mass m does not appear in the ODE). It is useful to notate $\omega = \sqrt{\frac{g}{L}}$.

(i) Calculate the solution $y(t)$ by initially elongating the pendulum by $y_0 > 0$ and then releasing it.
(ii) Calculate the velocity $y'(t)$ (radiants/sec) of the pendulum when it passes the vertical.
(iii) The *frequency* $f = \frac{\beta}{2\pi}$ of $\sin(\beta t)$ (or $\cos(\beta t)$) is the number of times per time unit the motion repeats; e.g. if $\beta = 2\pi$ the frequency is $f = 1$; if $\beta = 5\pi$ the frequency is $f = 2.5$). Calculate the frequency of the pendulum in terms of g and L.
(iv) Complete the sentence: *If one doubles the length of the pendulum, the frequency .....*
(v) If you observe a frequency of $f = 5$, how long is the pendulum?

Problem 2. Consider a spring with spring constant $k = 9$ on which we attach a mass $m = 1\,kg$. We pull the mass downwards from its resting position by $10\,cm$ and release it. The ODE for this situation is

$$y'' + \gamma y' + 9y = 0$$

where γ is the friction coefficient.

(i) Calculate the motion $y(t)$ of the mass when the friction coefficient is $\gamma = 10$, $\gamma = 6$ and $\gamma = \sqrt{20}$. Characterize in each case whether the motion is *underdamped* (two distinct real roots of the characteristic polynomial); *overdamped* (complex conjugate roots), *critically damped* (double root), and draw a reasonably accurate graph of your solutions (including units and labels on the axes).
(ii) Calculate in each of the cases above how long it takes for the motion to have died down to an amplitude of less than $10^{-2}cm$.
(iii)
(iv) What is the frequency of the motion in the underdamped case?
(v) Now insert the mass in some liquid medium and assume you observe oscillatory movement of frequency $f = \frac{\sqrt{5}}{2\pi}$. Determine the friction coefficient γ for this case?

Problem 3. Consider the ODE $y'' + 4y' + 5y = 0$. Determine the solution which satisfies the initial condition $y(0) = 1$ and $y'(0) = 2$ and try to write it in the form $Ae^{\alpha t}\sin(\beta t - \delta)$.

Problem 4. Consider the *inhomogeneous* ODE $y'' - 3y' + 2y = 1$.

(i) Guess one solution of this ODE (Hint: what are the constant solutions $y(t) = constant$?)
(ii) Find all solutions of the corresponding homogeneous ODE.
(iii) Find all solutions of the inhomogeneous ODE (recall the inhomogeneous super-position mantra: **all** *solutions of the inhomogeneous linear ODE are given by* **one** *solution of the inhomogeneous ODE plus* **all** *solutions of the homogeneous ODE.* What happens to those solutions as $t \to \infty$?
(iv) Find the solution with initial condition $y(0) = y'(0) = 0$.

Problem 5. Write the following complex numbers z in standard form $x + iy$:

$$z = e^{2+i}\,, \quad z = e^{i\pi}\,, \quad z = e^{-2+\frac{\pi i}{2}}\,, \quad z = 2^i$$

For the last recall the identity $a^b = e^{b \ln a}$ for change of an arbitrary base to base e.

Problem 6. A swing door of mass $m = 10\,kg$ on a hinge (modeled on a spring with spring constant $k = 10$ and friction $\gamma = 20$) is kicked open by an initial speed of $3\,m/sec$. The ODE for this problem is

$$my'' + \gamma y' + ky = 0$$

where $y(t)$ measures how far out the door swings at time t.

(i) Find the solution $y(t)$ of this problem.
(ii) How much does the door maximally open and how long does it take for that to happen?
(iii) Draw an accurate (labelled) graph of your solution $y(t)$ and its velocity $y'(t)$, and describe in words the position and velocity of the door motion.
(iv) How long does it take for the door to come within one centimeter of being closed?
(v) Will the door ever come back to its closed position?