HOMEWORK 9, M 331.2
DUE 11/30/16

Please hand in your home work before class, have it neatly written, organized (the grader will not decipher your notes), stapled, with your name and student ID on top.

Problem 1. Consider the inhomogeneous ODE

$$y'' - \frac{2}{t^2} y = \ln t$$

(i) Verify that $y_1(t) = t^2$ and $y_2(t) = -\frac{1}{3t}$ are solutions of the corresponding homogeneous ODE $y'' - \frac{2}{t^2} = 0$.
(ii) Verify that $y_1(t)$ and $y_2(t)$ are independent solutions by calculating their Wronskian $W(y_1, y_2)$.
(iii) Find a particular solution $y_P(t)$ of the inhomogeneous ODE.
(iv) Find the solution of the inhomogeneous ODE satisfying the initial conditions $y(1) = 1$ and $y'(1) = 0$.

Problem 2. You are holding a spring with spring constant $k = 16$ on one end and attach a mass $m = 1\,kg$ on the other end. Now you start moving your hand up and down in a periodic motion thereby exerting a periodic force on the mass-spring system. The resulting ODE and initial conditions for the position $y(t)$ of the mass (measured from equilibrium) are

$$y'' + 16y = 8\cos(\omega t) \qquad y(0) = y'(0) = 0$$

where ω is (2π times) the frequency of your hand motion.

(i) Calculate the motion $y(t)$ of the mass for arbitrary choice of ω, assuming that by moving your hand you will never hit exactly the resonance frequency $\omega = 4$. To obtain a nice formula in the end you may want to use the identity $\cos(x) - \cos(y) = 2\sin(\frac{x+y}{2})\sin(\frac{y-x}{2})$.
(ii) How does the amplitude of the motion of the mass depend on ω, i.e. what is the largest oscillation in $y(t)$? Write down an explicit formula.
(iii) Near which value should ω be to achieve very large amplitudes of the motion of the mass?
(iv) If you move your hand very rapidly up and down (that is, ω is very large), what happens to the amplitude of $y(t)$?
(v) Finally calculate the solution for the case when your hand moves with frequency exactly equal to the eigenfrequency, i.e. $\omega = 4$.

Problem 3. A 1 kilogram cube sliding on a horizontal surface oscillates on the end of a spring. The extreme displacement of the mass as it oscillates is 0.1 m and its period of oscillation is 0.5 seconds. After 16 periods, the cube comes within the fraction of e^{-2} of its extreme displacement. Determine the spring constant and friction coefficient.

Problem 4. Consider the inhomogeneous ODE

$$y'' + 6y' + 10y = f(t)\,.$$

(i) Find the solution of the ODE with inhomogeneity $f(t) = e^{3t}$ subject to the initial condition $y(0) = 1$, $y'(0) = 3$.

(ii) Find the solution of the ODE with inhomogeneity $f(t) = \sin t$ satisfying $y(0) = y'(0) = 0$.
(iii) Find a particular solution when $f(t) = e^{-3t} \sin t$.
(iv) Find a particular solution of the ODE with inhomogeneity $f(t) = t + e^{3t}$.

Problem 5. Find *all solutions* of the inhomogeneous ODE

$$y'' + 2y' + y = e^t .$$

Problem 6. Find the solution of the inhomogeneous ODE

$$y'' + 3y' + 2y = e^{-2t}$$

with initial conditions $y(0) = y'(0) = 0$.

Problem 7. Find a particular solution of the inhomogeneous ODE

$$y'' + 6y' + 10y = (1 + 2t)e^{-3t} \cos(t)$$

Problem 8. Consider the harmonic oscillator given by

$$y'' + 2y' + 5y$$

We want to use the oscillator to amplify the amplitude of an incoming periodic signal $f(t) = 3\cos(\omega t)$ by a factor of 20 (very old radios more or less worked like that, so do loud speakers – in both one has an electric circuit – but this basic idea is prevalent in modern devices, On an old radio the dials would change the values 2 and 5 of the oscillator so that different incoming frequencies, stations, could be amplified). Question is, for the given oscillator, what frequency ω do we have to choose. Mathematically this means that we have to solve the ODE

$$y'' + 2y' + 5y = 3\cos(\omega t)$$

with initial data $y(0) = y'(0) = 0$ (the oscillator is initially at rest) and choose ω so that the solution has amplitude 60 (since the incoming signal has amplitude 3).

(i) Calculate the solution $y(t)$ for arbitrary values of ω.
(ii) How many seconds does it take for the homogenous part of the solution to decrease to an amplitude of 1/50?
(iii) For our amplification problem, we take the view point that we wait till the homogenous part of the solution is negligible, so that we only care about the particular solution $y_P(t)$. Write $y_P(t)$ in the form $A\cos(\omega t - \delta)$ and give an explicit formula for A in terms of ω.
(iv) For which value of ω is the amplitude of $y_P(t)$ maximal and how large is it? Draw a graph of A as a function of ω.
(v) Find the value of ω so that $A = 60$.