Problem 1. The Frenet frame of a curve in $\mathbb{R}^3$. For a regular plane curve (and more generally for a regular curve on a 2-dimensional surface - e.g. the 2-sphere above) we could construct a unique adapted frame $F$. This is not the case for curves in higher dimensional spaces. Besides the curve being regular we need more conditions to ensure the existence of a unique adapted frame, which then will give invariants of the curve, which in turn reconstruct the curve up to Euclidean motions.

Let $\gamma: I \rightarrow \mathbb{R}^3$ be an arclength parametrized curve. Then $T = \gamma'$ has unit length.

(i) Show that $\langle \gamma'', T \rangle = 0$. Thus, provided that $\gamma''$ is nowhere vanishing, we can define $N := \gamma''/||\gamma''||$ and obtain a moving basis $\{T, N, T \times N\}$. A regular space curve for which $\gamma''$ is nowhere vanishing is called a Frenet curve. From now on we assume that we have this property.

(ii) Let $\gamma$ be an arclength parametrized Frenet curve. Define the curvature function to be $\kappa := ||\gamma''|| > 0$ and the torsion function $\tau := \langle T \times N, N' \rangle$. Show that the adapted frame $F = (T, N, T \times N): I \rightarrow \text{SO}(3, \mathbb{R})$ and calculate $A = F^{-1}F'$ in terms of $\kappa$ and $\tau$. This is just a more compact formulation of the following: calculate $T'$ and express it in the basis $\{T, N, T \times N\}$; then do the same for $N'$ and $(T \times N)'$. You can save yourself computations by using that $\{T, N, T \times N\}$ is an orthonormal basis for each $t \in I$ and look up HW 5 Problem 1.

(iii) If $\gamma$ is an arclength parametrized planar curve, we can regard it as a space curve. Show that this space curve has $\tau \equiv 0$. Also prove the converse: if a space curve has $\tau \equiv 0$ then it lies in some plane $P \subset \mathbb{R}^3$.

(iv) Show that given functions $\kappa: I \rightarrow \mathbb{R}$, $\kappa(t) > 0$ for all $t \in I$, and $\tau: I \rightarrow \mathbb{R}$ both smooth, there exists a unique (up to Euclidean motion) Frenet curve in $\mathbb{R}^3$ whose curvature and torsion are $\kappa$ and $\tau$ respectively.

(v) Classify all the Frenet space curves which have curvature and torsion constant.

Problem 2. Let $\gamma: I \rightarrow S^2$ be a regular curve in the 2-sphere. We have discussed (in HW 5 Problem 2) its curvature, which we shall call for now geodesic curvature (to not get confused with the curvature this curve may have as a curve in $\mathbb{R}^3$) and label it as $\kappa_g$.

(i) Regarding the curve $\gamma$ as a space curve $S^2 \subset \mathbb{R}^3$, and assuming it to be Frenet, calculate its curvature $\kappa$ and torsion $\tau$ in terms of $\kappa_g$.

(ii) State and prove necessary and sufficient conditions for a Frenet curve $\gamma: I \rightarrow \mathbb{R}^3$ to be contained in some 2-sphere $S = \{x \in \mathbb{R}^3; ||x - c|| = r\}$ of center $c$ and radius $r > 0$.

Problem 3. Find all the critical points (under compactly supported variations) for the length functional

$$L(\gamma) = \int_I ||\gamma'(t)|| dt$$

on regular curves $\gamma: I \rightarrow \mathbb{R}^n$ in any dimension $n$. Since there is no notion of curvature for $n \geq 4$ (at least we did not discuss that) you will just have to leave the term $\gamma''$ in the variational calculus and try to find the conditions $\gamma$ has to satisfy to be a critical point.
Problem 4. Consider the energy functional

\[ E(\gamma) = \int_{I} ||\gamma'(t)||^2 dt \]
on regular curves \( \gamma: I \to S^2 \).

(i) Show that if \( \gamma_s: I \to S^2 \) is any variation of \( \gamma = \gamma_0 \) by curves on \( S^2 \) then its variational vector field \( V = \dot{\gamma} \) satisfies \( \langle V, \gamma \rangle = 0 \).

(ii) Let \( V: I \to \mathbb{R}^3 \) be smooth with \( \langle V, \gamma \rangle = 0 \). Show that there is a variation \( \gamma_s \) of \( \gamma \) by curves on \( S^2 \). Moreover, if \( V \) is compactly supported so will be \( \gamma_s \).

(iii) Characterize the critical points of \( E \) under compactly supported variations \( \gamma_s \) by curves on \( S^2 \).

(iv) If the curve \( \gamma \) were simply closed, it would define an enclosed area (modulo \( 4\pi \) – one has two areas for a simply closed curve on the 2-sphere). Characterize area constrained \( E \) critical curves \( \gamma \). Using the Lagrange multiplier approach what is the functional \( \tilde{E} = E + \lambda \) you will work with? You will need to find an expression of the enclosed area in terms of \( \gamma \).