Midterm, Differential Geometry

due 3/24/17

Problem 1. The Frenet frame of a curve in $\mathbb{R}^3$. For a regular plane curve (and more generally for a regular curve on a 2-dimensional surface - e.g. the 2-sphere above) we could construct a unique adapted frame F. This is not the case for curves in higher dimensional spaces. Besides the curve being regular we need more conditions to ensure the existence of a unique adapted frame, which then will give invariants of the curve, which in turn reconstruct the curve up to Euclidean motions.

Let $\gamma\colon I \to \mathbb{R}^3$ be an arclength parametrized curve. Then $T = \gamma'$ has unit length.

(i) Show that $< \gamma'', T >= 0$. Thus, provided that γ'' is nowhere vanishing, we can define $N := \gamma''/||\gamma''||$ and obtain a moving basis $\{T, N, T \times N\}$. A regular space curve for which γ'' is nowhere vanishing is called a *Frenet curve*. From now on we assume that we have this property.

(ii) Let γ be an arclength parametrized Frenet curve. Define the curvature function to be $\kappa := ||\gamma''|| > 0$ and the *torsion* function $\tau :=< T \times N, N' >$. Show that the adapted frame $F = (T, N, T \times N)\colon I \to \mathbf{SO}(3, \mathbb{R})$ and calculate $A = F^{-1}F'$ in terms of κ and τ. This is just a more compact formulation of the following: calculate T' and express it in the basis $\{T, N, T \times N\}$; then do the same for N' and $(T \times N)'$. You can save yourself computations by using that $\{T, N, T \times N\}$ is an orthonormal basis for each $t \in I$ and look up HW 5 Problem 1.

(iii) If γ is an arclength parametrized planar curve, we can regard it as a space curve. Show that this space curve has $\tau \equiv 0$. Also prove the converse: if a space curve has $\tau \equiv 0$ then it lies in some plane $P \subset \mathbb{R}^3$.

(iv) Show that given functions $\kappa\colon I \to \mathbb{R}$, $\kappa(t) > 0$ for all $t \in I$, and $\tau\colon I \to \mathbb{R}$ both smooth, there exists a unique (up to Euclidean motion) Frenet curve in $\mathbb{R}^3$ whose curvature and torsion are κ and τ respectively.

(v) Classify all the Frenet space curves which have curvature and torsion constant.

Problem 2. Let $\gamma\colon I \to S^2$ be a regular curve in the 2-sphere. We have discussed (in HW 5 Problem 2) its curvature, which we shall call for now *geodesic curvature* (to not get confused with the curvature this curve may have as a curve in $\mathbb{R}^3$) and label it as κ_g.

(i) Regarding the curve γ as a space curve $S^2 \subset \mathbb{R}^3$, and assuming it to be Frenet, calculate its curvature κ and torsion τ in terms of κ_g.

(ii) State and prove necessary and sufficient conditions for a Frenet curve $\gamma\colon I \to \mathbb{R}^3$ to be contained in some 2-sphere $S = \{x \in \mathbb{R}^3\,;\, ||x - c|| = r\}$ of center c and radius $r > 0$.

Problem 3. Find all the critical points (under compactly supported variations) for the length functional

$$L(\gamma) = \int_I ||\gamma'(t)||dt$$

on regular curves $\gamma\colon I \to \mathbb{R}^n$ in any dimension n. Since there is no notion of curvature for $n \geq 4$ (at least we did not discuss that) you will just have to leave the term γ'' in the variational calculus and try to find the conditions γ has to satisfy to be a critical point.

Problem 4. Consider the energy functional

$$E(\gamma) = \int_I ||\gamma'(t)||^2 dt$$

on regular curves $\gamma \colon I \to S^2$.

(i) Show that if $\gamma_s \colon I \to S^2$ is any variation of $\gamma = \gamma_0$ by curves on S^2 then its variational vector field $V = \dot{\gamma}$ satisfies $< V, \gamma >= 0$.

(ii) Let $V \colon I \to \mathbb{R}^3$ be smooth with $< V, \gamma >= 0$. Show that there is a variation γ_s of γ by curves on S^2. Moreover, if V is compactly supported so will be γ_s.

(iii) Characterize the critical points of E under compactly supported variations γ_s by curves on S^2.

(iv) If the curve γ were simply closed, it would define an enclosed area (modulo 4π – one has two areas for a simply closed curve on the 2-sphere). Characterize area constrained E critical curves γ. Using the Lagrange multiplier approach what is the functional $\tilde{E} = E+?$ you will work with? You will need to find an expression of the enclosed area in terms of γ.