Homework 1, M 523

Problem 1. Let G be a group. Show that the neutral element e is unique, that is, if there were two neutral elements e_1 and e_2, then $e_1 = e_2$.

Problem 2. Let G be a group. Show that the inverse element g^{-1} of an element $g \in G$ is unique, that is, if h_1 and h_2 are two inverse elements of g, then $h_1 = h_2$.

Problem 3. Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ given by the rule $f(x) = x^2$. Is this function injective? Prove your answer. Can you make it injective by changing the domain? Which domain? Is the function surjective? Can you prove your answer? Find a domain and codomain so that f becomes bijective and calculate the inverse function f^{-1}.

Problem 4. Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = \sin(x)$. Is this function injective? Prove your answer. Can you make it injective by changing the domain? Which domain? Is the function surjective? Can you prove your answer? Find a domain and codomain so that f becomes bijective.

Problem 5. Show that $\sqrt{3}$ is not a rational number. Show that for any prime number p the square root $\sqrt{p}$ is not rational.

Problem 6. Use Newton's method to find a zero of an equation to obtain a recursively defined sequence of rational numbers x_n, $n \in \mathbb{N}$, which, if it "converged", would "converge" to $\sqrt{2}$. Calculate the first 6 terms of your sequence.

Problem 7. Use the principle of induction to verify:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=0}^{n} q^k = \frac{1-q^{n+1}}{1-q}$$

How do you interpret this equation when $q = 1$? Give also a direct proof of the latter equation.