Homework 2, M 523

Problem 1. Consider the real numbers with the strict order $<$ satisfying the order properties (O1) – (O4). Show:

(i) $x < y$ if and only if $y - x > 0$.
(ii) $x < y$ if and only if $-y < -x$.
(iii) If $x \neq 0$ then $x^2 > 0$.

Problem 2. Show that every nonempty subset $A \subset \mathbb{N}$ of the natural numbers has a smallest element $n_0 \in A$, i.e., $n \geq n_0$ for all $n \in A$.

Problem 3. Show that the strict order $<$ on $\mathbb{Q}$ is *archimedian*, that is, for every $r \in \mathbb{Q}$ there is an $n \in \mathbb{N}$ so that $r < n$.

Problem 4. Show that arbitrarily close to any rational number there is a real (non-rational) number. In other words, show that to each real $\epsilon > 0$ and each rational $r \in \mathbb{Q}$ there exists $x \in \mathbb{R} \setminus \mathbb{Q}$ with $|x - r| < \epsilon$.

Problem 5. We call a set X countable if there is a bijection $f \colon X \to \mathbb{N}$. Show:

(i) If X and Y are countable then also their union $X \cup Y$ is countable.
(ii) $\mathbb{Z}$ is countable.
(iii) $\mathbb{Q}$ is countable.

Problem 6. Use induction to show

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ and $k! = 1 \cdot 2 \cdots k$.

Problem 7. Prove Bernoulli's inequality $(1+x)^n \geq 1+nx$ for $x \geq -1$.

Problem 8. Show that for a sequence (x_n) the following are true:

(i) $\lim x_n = 0$ if and only if $\lim |x_n| = 0$.
(ii) $\lim x_n = L$ implies $\lim |x_n| = |L|$. Is the converse true? Prove or give a counterexample.
(iii) $\lim x_n = L$ if and only if $\lim |x_n - L| = 0$.