Homework 3, M 523

Problem 1. Let $a > 0$ be a real number and consider the sequence (x_n) given by

$$x_{n+1} = \frac{x_n^2 + a}{2x_n}$$

which is suggested from Newton's method to find a positive root of the quadratic polynomial $f(x) = x^2 - a$. Our aim is to show that this sequence converges and to calculate its limit. Show the following:

(i) if $0 \neq x_1 \in \mathbb{Q}$ is rational so are all x_n. In this case our sequence is a sequence of rational numbers.

(ii) Show that for any starting value $x_1 > 0$ all sequence elements are positive.

(iii) Show that for any choice of $x_1 \neq 0$ the sequence satisfies $x_n^2 > a$ for all $n \geq 2$.

(iv) Show that if $x_1 > 0$ the sequence is monotone decreasing for $n \geq 2$ and thus (why?) convergent by a Theorem we proved in class.

(v) Show that $(\lim x_n)^2 = a$ and therefore $\lim x_n$ is a positive square root of a (provided we started with a positive $x_1 > 0$). Notice that we have shown the existence of all square roots of positive real numbers and moreover those square roots can be obtained as limits of sequences of rational numbers.

Problem 2. Since we now know that every real number $a \geq 0$ has a positive square root $\sqrt{a}$ show that the (positive) square root is monoton: $0 \leq a \leq b$ if and only if $\sqrt{a} \leq \sqrt{b}$.

Problem 3. Let (x_n) be a sequence of *non-negative* real numbers. Show that $\lim x_n = L$ if and only if $\lim(x_n^2) = L^2$. Is this also true for arbitrary convergent sequences? Which direction holds in general?

Problem 4. Show that for $0 \leq q < 1$ the sequence $x_n = q^n$ converges. Calculate its limit.

Problem 5. Consider the sequence $x_n := \sum_{k=0}^{n} q^k$ for a real number q. For which numbers q does (x_n) converge/diverge? Calculate the limit in the converging case.

Problem 6. Let (x_n) be a sequence such that for some $N \in \mathbb{N}$ we have $|x_{n+1} - x_n| < (1/2)^n$ for all $n \geq N$. Show that (x_n) is Cauchy.

Problem 7. Approach the first problem differently by showing that the sequence defined by $x_{n+1} = \frac{x_n^2+a}{2x_n}$ is in fact a Cauchy sequence (rather than showing it is a monotone and bounded sequence) and therefore convergent.

Problem 8. Let (x_n) be a sequence of real numbers satisfying $x_n < M$ for all $n \in \mathbb{N}$. Assuming (x_n) converges, show that $\lim x_n \leq M$. Can you give an example where under our assumptions $\lim x_n = M$? So despite $x_n < M$ for all $n \in \mathbb{N}$ the limit of (x_n) can be equal to M.