Homework 4, M 523

Problem 1. Show that the existence of sup/inf for bounded above/below non-empty subsets of $\mathbb{R}$ implies that every Cauchy sequence is convergent in $\mathbb{R}$.

Problem 2. Show that knowing that bounded monotone sequences converge in $\mathbb{R}$ implies that every bounded above/below non-empty subset of $\mathbb{R}$ has a sup/inf.

Thus we have shown that convergency of Cauchy sequences, bounded monotonicity convergency and existence of sup/inf for bounded subsets are all equivalent in $\mathbb{R}$.

Problem 3. Let $A_n := [a_n, b_n] \subset \mathbb{R}$ be a sequence of nested closed intervals, $a_n < b_n$, and $A_{n+1} \subset A_n$. Show that the infinite intersection $A_1 \cap A_2 \cap A_3 \cap A_4 \cdots$ is non-empty. Is this true if the intervals A_n are not closed, e.g. for open intervals?

Problem 4. Some calculations with complex numbers:

(i) $(1-3i)^{-1} =?$
(ii) $i^n =?$ for $n \in \mathbb{Z}$.
(iii) z and w in $\mathbb{C}$ are perpendicular vectors in $\mathbb{R}^2$ if and only if $\mathrm{Re}(z\bar{w}) = 0$.
(iv) z and w in $\mathbb{C}$ are linearly dependent vectors in $\mathbb{R}^2$ if and only if $\mathrm{Im}(z\bar{w}) = 0$.
(v) Proof Pythagoras' Theorem: if z and w are perpendicular then $|z|^2 + |w|^2 = |z-w|^2$

Problem 5. Let $\sum_{k=1}^{\infty} z_k$ be an infinite series. Then $\sum_{k=1}^{\infty} z_k$ converges to L if and only if $\sum_{k=1}^{\infty} \mathrm{Re}\, z_k$ converges to $\mathrm{Re}\, L$ and $\sum_{k=1}^{\infty} \mathrm{Im}\, z_k$ converges to $\mathrm{Im}\, L$. In other words:

$$\mathrm{Re} \sum_{k=1}^{\infty} z_k = \sum_{k=1}^{\infty} \mathrm{Re}\, z_k \quad \text{and} \quad \mathrm{Im} \sum_{k=1}^{\infty} z_k = \sum_{k=1}^{\infty} \mathrm{Im}\, z_k \,.$$

Furthermore (or perhaps first) show that

$$\overline{\sum_{k=1}^{\infty} z_k} = \sum_{k=1}^{\infty} \overline{z_k} \,.$$

Problem 6. Show that the series $\sum_{k=2}^{\infty} \frac{1}{k^2-1}$ converges and calculate its limit. Hint: write out the partial sums; think partial fractions.

Problem 7. Show that the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges (you probably will not be able to calculate its limit). Hint: comparison test.