Problem 1. Consider the series $\sum_{k=0}^{\infty}(-1)^k \frac{1}{\sqrt{k+1}}$ which converges by the Leibniz criterion. Show that the Cauchy product of this series with itself is divergent. (Hint: estimate the "inside" finite sum in the Cauchy product uniformly away from zero).

Problem 2. Show that the Cuachy product of the two divergent series

$$(2 + 2 + 2^3 + 3^4 + 2^5 + \cdots)(-1 + 1 - 1 + 1 - 1 + 1 \cdots)$$

is absolut convergent.

Problem 3. Prove the functional relation of the exponential function:

$$\exp(z + w) = \exp(z)\exp(w)$$

for $z, w \in \mathbb{C}$.

Problem 4. Show the following properties of the exponential function:
(i) $\exp(z) \neq 0$ for all $z \in \mathbb{C}$.
(ii) $\exp(0) = 1$.
(iii) $\exp(-z) = \frac{1}{\exp(z)}$

Problem 5. Show that the "infinite triangle inequality"

$$|\sum_{k=1}^{\infty} z_k| \leq \sum_{k=1}^{\infty} |z_k|$$

holds (independent of convergency).

Problem 6. Sometime the elements of an infinite series $\sum_{k=0}^{\infty} z_k(n)$ depend on a parameter $n \in \mathbb{N}_0$ and we want to calculate $\lim_{n\to\infty} \sum_{k=0}^{\infty} z_k(n)$ by interchanging the limit with the sum.
(i) Find an example where

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} z_k(n) \neq \sum_{k=0}^{\infty} \lim_{n\to\infty} z_k(n)\,.$$

(ii) Show that the assumptions $z_k(n+1) \geq z_k(n) \geq 0$ for all $k, n \in \mathbb{N}_0$ and $\sum_{k=0}^{n} z_k(n) \leq B$ for all $n \in \mathbb{N}_0$ imply convergency of the series and that the limit can be interchanged with the sum. (Hint: consider $a_{k,l} := z_k(l) - z_k(l-1)$ so that $\sum_{l=0}^{n} a_{k,l} = z_k(n)$ and try to apply the theorem about summation of an infinite matrix array).

Problem 7. Show that the sequence $x_n = (1 + 1/n)^n$ converges to *Euler's number* $e := \exp(1)$. Use both the series and the sequence to calculate an approximate value for e. Which method works better? Can you give an error estimate, i.e., how far to you have to compute to be sure that your value is correct up to say 5 decimal points (without of course knowing the precise value for e).