HOMEWORK 6, M 523

Problem 1. Let $f\colon D \to \mathbb{C}$ be a function on a domain $D \subset \mathbb{C}$. Show:

(i) The functions $z \mapsto \operatorname{Re} z$, $z \mapsto \operatorname{Im} z$, and $z \mapsto \bar{z}$ are continuous.
(ii) f is continuous if and only if $\operatorname{Re} f$ and $\operatorname{Im} f$ are continuous real valued functions on D.
(iii) f is continuous if and only if $\bar{f}$ is continuous (where $\bar{f}(z) := \overline{f(z)}$).

Problem 2. Define the general exponential function to arbitrary base $a > 0$ and exponent $z \in \mathbb{C}$ via

$$a^z := \exp(z \log a)\,.$$

Show the following properties:

(i) a^z is a continuous function on $\mathbb{C}$;
(ii) $a^n = a \cdot a \cdots a$ (n-times) for $n \in \mathbb{N}$;
(iii) $a^{n/m} = \sqrt[m]{a^n}$ for $n \in \mathbb{N}$ and $m \in \mathbb{N}$;
(iv) $e^z = \exp(z)$ for $e = \exp(1)$;
(v) $a^{z+w} = a^z a^w$ for $z, w \in \mathbb{C}$ and thus $a^{-z} = 1/a^z$.

Problem 3. (i) Show that $B_\epsilon(z_0) := \{z \in \mathbb{C}\,;\, |z - z_0| < \epsilon\}$ is an open subset of $\mathbb{C}$.
(ii) Show that a subset $D \subset \mathbb{C}$ is open (resp. closed), if and only if $D^\circ = D$ (resp. $\bar{D} = D$).
(iii) Calculate the closures of $\mathbb{Q} \subset \mathbb{R}$, $\mathbb{R}\backslash\mathbb{Q} \subset \mathbb{R}$, $\{1/n\,;\, n \in \mathbb{N}\} \subset \mathbb{R}$.
(iv) What is $\mathbb{Q}^\circ \subset \mathbb{R}$?

Problem 4. Show that a uniformly continuous function $f\colon D \to \mathbb{C}$ maps Cauchy sequences in D to Cauchy sequences. Is the converse true, that is, if a function has this property, is it uniformly continuous?

Problem 5. Prove or disprove:

(i) $f(x) = 1/x$ is uniformly continuous on $(0, 1]$;
(ii) $f(x) = x^2$ is uniformly continuous on $[0, \infty)$;
(iii) $f(x) = \sqrt{x}$ is uniformly continuous on $(0, 1)$.

Problem 6. Let $D \subset \mathbb{C}$. We call a set $U \subset D$ open, if $U = D \cap O$ with $O \subset \mathbb{C}$ open. Show that $f\colon D \to \mathbb{C}$ is continuous if and only if pre-images of open subsets in $\mathbb{C}$ are open in D.