Homework 7, M 523

Problem 1. Let $f\colon [0,1] \to [0,1]$ be a continuous map. Show that f has a fixed point, that is there exists $x_0 \in [0,1]$ so that $f(x_0) = x_0$. Can there be many fixed points or is a fixed point unique?

Problem 2. Consider the following functions on $\mathbb{R} \setminus \{0\}$:

$$\frac{\sin x}{x}\,, \qquad x\sin(1/x)\,, \qquad x^2\sin(1/x)$$

Can you extend these function to continuous functions on $\mathbb{R}$ and if so, how? Where are the extended functions differentiable?

Recall, that a *normed vector space* is a vector space V (over $\mathbb{R}$ or $\mathbb{C}$) together with a function $|| \ ||\colon V \to \mathbb{R}$ satisfying

(i) $||x|| \geq 0$ for all $x \in V$ and $||x|| = 0$ if and only of $x = 0$.
(ii) $||\lambda x|| = |\lambda|||x||$ for all $x \in V$ and all scalars λ.
(iii) $||x + y|| \leq ||x|| + ||y||$ or all $x, y \in V$.

A complete (i.e., Cauchy sequences in V converge in V) normed vector space is called a *Banach space*.

Problem 3. Let $(V, || \ ||)$ be a normed vector space. Show that the norm function $|| \ ||\colon V \to \mathbb{R}$ is continuous: if $x_n \to x$ in V (that is $||x_n - x|| \to 0$) then $||x_n|| \to ||x||$ in $\mathbb{R}$. Conclude that "spheres" $S_r(x_0) := \{x \in V\,;\, ||x - x_0|| = r\}$ of radius $r > 0$ centered at $x_0 \in V$ and closed balls $\bar{B}_r(x_0) := \{x \in V\,;\, ||x - x_0|| \leq r\}$ are closed subsets of V.

Problem 4 (Banach Fixed Point Theorem)**.** Let $(V, || \ ||)$ be a Banach space, $B \subset V$ closed (e.g. B could be a sphere or a closed ball), and $T\colon B \to B$ a contracting map, that is $||T(x) - T(y)|| \leq M||x - y||$ for $0 \leq M < 1$. Show:

(i) If T has a fixed point, that is a point $x_0 \in B$ such that $T(x_0) = x_0$, then x_0 is unique.
(ii) T always has a fixed point, and thus with (i) always has a unique fixed point in B.

Hint: first part by contradiction. For the second part carry out an iteration: start with any $x_1 \in B$ and set $x_2 := T(x_1)$ and continue this way to generate a sequence $(x_n)_{n \in \mathbb{N}}$ in B. Use the contraction property to show the sequence is Cauchy.... Using (i) you see that it didn't matter where you started in the iteration, you always converge to the same fixed point.

Problem 5. Let $V = C^0(D, \mathbb{C})$ be the vector space of (bounded, otherwise the norm is not defined for all $f \in V$) continuous functions on a non-empty open $D \subset \mathbb{C}$ with the supremum norm $||f|| := \sup_{z \in D} |f(z)|$.

(i) Show that the vectors $f_n := z^n$, $n \in \mathbb{N}_0$, are linearly independent in V.

(ii) Let $D = \mathbb{R}$. Show that the vectors $f_n := e^{inx}$, $n \in \mathbb{Z}$, are linearly independent in V.

(iii) The unit sphere $S_1(0) \subset V$ is closed and bounded (why?). Can you find a sequence $f_n \in S_1(0)$ which does not have a convergent subsequence? In the light of Bolzano-Weierstrass this indicates that in infinite dimensions compactness may need to be defined differently.

(iv) Let $D = [0, 1]$ and consider the sequence of functions $f_n \in V$

$$f_n(x) = \begin{cases} 2nx & 0 \le x \le 1/2n \\ -2nx + 2 & 1/2n \le x \le 1/n \\ 0 & \text{else} \end{cases}$$

Is this sequence Cauchy? Is there a limit function? Does the sequence converge uniformly to the limit function?