Midterm Exam, M 523

You need to have 4 problems essentially correct to receive an A.

Problem 1. Show that every real number $x \in \mathbb{R}$ has a decimal representation $x = x_n x_{n-1} \dots x_0 \,.\, x_{-1} x_{-2} \dots$ with integers $0 \leq x_k \leq 9$. Conversely, every such expression defines a real number.

Problem 2. Recall Euler's number $e := \exp(1) = \sum_{k=0}^{\infty} \frac{1}{k!}$. Show that e is not a rational number.

Problem 3. Prove or disprove:

(i) Let $a_n \geq 0$ with $\sum_{n=1}^{\infty} a_n$ convergent, then $\sum_{n=1}^{\infty} \sqrt{a_n}$ converges.

(ii) Let $a_n \geq 0$ with $\sum_{n=1}^{\infty} a_n$ convergent, then $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$ converges.

Problem 4. Prove or disprove:

(i) Let $f \colon \mathbb{R} \to \mathbb{R}$ be the function assigning rational numbers to 1 and irrational numbers to 0, that is, $f(x) = 1$ for $x \in \mathbb{Q}$ and $f(x) = 0$ for $x \in \mathbb{R} \backslash \mathbb{Q}$. Then f is continuous.

(ii) Let $f \colon D \to \mathbb{C}$ be continuous on $D \subset \mathbb{C}$ and let $(z_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in D. Then $(f(z_n))_{n \in \mathbb{N}}$ is a Cauchy sequence.

Problem 5. Show that the exponential function $\exp \colon \mathbb{R} \to (0, \infty)$ is strictly increasing and surjective and thus a bijective map.