PROBLEM SHEET 4, MATH 523

Problem 1. Consider the function $f : \mathbb{R}_+ \to \mathbb{R}_+$ given by $f(x) = x^p$ where $p \in \mathbb{Q}$. Show:

1. $f(ab) = f(a)f(b)$
2. f is strictly monoton increasing if $p > 0$ and strictly monoton decreasing if $p < 0$.
3. f is bijective. Determine the inverse function $f^{-1} : R_+ \to \mathbb{R}_+$.

Problem 2. Show that $S^1 = \{z \in \mathbb{C}\,;\, |z| = 1\}$ is a commutative group. Verify the alternate description $S^1 = \{z \in \mathbb{C}\,;\, z^{-1} = \bar{z}\}$.

Problem 3. Draw the following sets:

1. $M_1 = \{z \in \mathbb{C}\,;\, |z-1| = |z+1|\}$,
2. $M_2 = \{z \in \mathbb{C}\,;\, 1 < |z-i| < 2\}$, and
3. $M_3 = \{z \in \mathbb{C}\,;\, |z| \geq 1, |\mathrm{Re}(z)| \leq \frac{1}{2}, \mathrm{Im}(z) > 0\}$.

Problem 4. The 5^{th}–roots of unity and the golden ratio (the Pythagorean order symbol and its irrationality):

1. Solve the equation $z^5 = 1$. Hint: $z^4+z^3+z^2+z+1 = (z^2+gz+1)(z^2-hz+1)$ where $g = h^{-1}$ are the golden ration introduced in problem sheet 2.1.
2. Let $\zeta = \frac{1}{2}(h+i\sqrt{4-h^2})$. Then the 5^{th}–roots of unity are ζ^n for $n = 1, \ldots, 5$. Moreover, $\zeta^2 = \overline{\zeta^3}$ and $\zeta^4 = \bar{\zeta}$.
3. The 5^{th}–roots of unity form the vertices of a regular 5-gon.
4. Show and interpret graphically $|\zeta^2 - 1| : |\zeta - 1| = g$.

Problem 5. Show that for all $z, w \in \mathbb{C}$ one has

1. $||z| - |w|| \leq |z - w|$, and
2. $|z+w|^2 + |z-w|^2 = 2(|z|^2 + |w|^2)$, the *parallelogram identity.*

Problem 6. Show that every circle and every line in $\mathbb{C}$ is described by the equation

$$a|z|^2 + 2\mathrm{Re}(bz) + c = 0 \qquad a, c \in \mathbb{R}\,, b \in \mathbb{C}\,, |b|^2 - ac > 0\,.$$

Conversely, the solutions to the above equation are circles and lines.

A *fractional linear transformation* of $\mathbb{C}$ is defined by

$$T : \mathbb{C} \setminus \{-\tfrac{d}{c}\} \to \mathbb{C}\,, \qquad T(z) = \tfrac{az+b}{cz+d}$$

with $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$.

We call a map $L : \mathbb{C} \to \mathbb{C}$ of the form $L(z) = az + b$, $a, b \in \mathbb{C}$ an *affine map.* Thus a fractional linear transformation is the quotient of 2 affine maps (analysts often call affine maps just linear maps, even if they are not linear in the sense of linear algebra).

Problem 7. Show that a fractional linear transformation (which is not affine) is uniquely determined by its values on 3 distinct points.

Problem 8. Show that every linear fractional transformation is either an affine map, or the composition $T = L_1 \circ I \circ L_2$ of affine maps L_k and the inversion $I(z) = \frac{1}{z}$.

Given 4 distinct points $z_1, \ldots, z_4 \in \mathbb{C}$ we define their *cross ratio*

$$[z_1, z_2, z_3, z_4] := \tfrac{z_2 - z_1}{z_3 - z_2} \cdot \tfrac{z_4 - z_3}{z_1 - z_4}\,.$$

Problem 9. Show that fractional linear transformations preserve the cross ratio. Hint: previous problem.

Problem 10. Show that a fractional linear transformation maps every circle (in its domain) into a circle.

Problem 11. Show that the cross ration of 4 distinct points is real, if and only if the 4 points lie on a circle. Hint: previous 2 problems and alternate characterization of S^1. Also, 3 distinct points determine a circle uniquely.