HOMEWORK 10, M 331
DUE 4/23/09

Problem 1. Find the solution of the inhomogeneous ODE

$$y'' + y = f(t)$$

with initial conditions $y(0) = y'(0) = 0$, where

$$f(t) = \begin{cases} 0 & t < \pi \\ 1 & \pi \le t < 3\pi \\ 0 & 3\pi \le t \end{cases}$$

First we rewrite f and a sum of Heaviside functions: $f(t) = u_\pi(t) - u_{3\pi}(t)$. The equation is then

$$y'' + y = u_\pi(t) - u_{3\pi}(t)$$

Taking the Laplace transform of both sides,

$$\begin{aligned}
\mathcal{L}[y'' + y] &= \mathcal{L}[u_\pi - u_{3\pi}] \\
\mathcal{L}[y''] + \mathcal{L}[y] &= \mathcal{L}[u_\pi] - \mathcal{L}[u_{3\pi}] \\
s^2\mathcal{L}[y] - sy(0) - y'(0) + \mathcal{L}[y] &= \frac{e^{-\pi s}}{s} - \frac{e^{-3\pi s}}{s} \\
(s^2+1)\mathcal{L}[y] &= \frac{e^{-\pi s}}{s} - \frac{e^{-3\pi s}}{s} \\
\mathcal{L}[y] &= \frac{e^{-\pi s}}{s(s^2+1)} - \frac{e^{-3\pi s}}{s(s^2+1)}
\end{aligned}$$

Taking the inverse Laplace transform of both sides,

$$\begin{aligned}
y &= \mathcal{L}^{-1}\left[\frac{e^{-\pi s}}{s(s^2+1)} - \frac{e^{-3\pi s}}{s(s^2+1)}\right] \\
&= \mathcal{L}^{-1}\left[\frac{e^{-\pi s}}{s(s^2+1)}\right] - \mathcal{L}^{-1}\left[\frac{e^{-3\pi s}}{s(s^2+1)}\right] \\
&= \mathcal{L}^{-1}\left[e^{-\pi s}\frac{1}{s(s^2+1)}\right] - \mathcal{L}^{-1}\left[e^{-3\pi s}\frac{1}{s(s^2+1)}\right]
\end{aligned}$$

First we will find $\mathcal{L}^{-1}\left[e^{-\pi s}\frac{1}{s(s^2+1)}\right]$. We use that $\mathcal{L}^{-1}[e^{-cs}F(s)] = u_c(t)f(t-c)$ where $f(t) = \mathcal{L}^{-1}[F]$. In this notation, we have $c = \pi$ and $F(s) = \frac{1}{s(s^2+1)}$. So, we need to find $\mathcal{L}^{-1}\left[\frac{1}{s(s^2+1)}\right]$. We use the method of partial fractions to rewrite F. The partial fractions decomposition is

$$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$$

Solving for the constants, we find $A = 1$, $B = -1$ and $C = 0$. Then

$$F(s) = \frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$$

Thus,

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}[F] \\ &= \mathcal{L}^{-1}\left[\frac{1}{s} - \frac{s}{s^2+1}\right] \\ &= \mathcal{L}^{-1}\left[\frac{1}{s}\right] - \mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right] \\ &= 1 - \cos t \end{aligned}$$

Then $\mathcal{L}^{-1}\left[e^{-\pi s}\frac{1}{s(s^2+1)}\right] = u_\pi(t)f(t-\pi) = u_\pi(t)[1-\cos(t-\pi)]$.

Similarly, $\mathcal{L}^{-1}\left[e^{-3\pi s}\frac{1}{s(s^2+1)}\right] = u_{3\pi}(t)f(t-3\pi) = u_{3\pi}(t)[1-\cos(t-3\pi)]$.

Putting these together,

$$\begin{aligned} y &= u_\pi(t)[1-\cos(t-\pi)] - u_{3\pi}(t)[1-\cos(t-3\pi)] \\ &= \begin{cases} 0 & t < \pi \\ 1-\cos(t-\pi) & \pi \le t < 3\pi \\ 0 & 3\pi \le t \end{cases} \end{aligned}$$

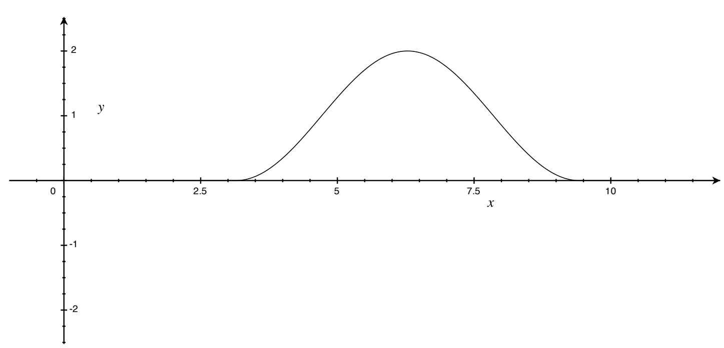

Figure 1

Problem 2. Find the solution of the inhomogeneous ODE

$$y'' + 3y' + 2y = u_1(t)$$

with initial conditions $y(0) = 0$ and $y'(0) = 1$.

Taking the Laplace transform of both sides,

$$\mathcal{L}[y'' + 3y' + 2y] = \mathcal{L}[u_1]$$

$$\mathcal{L}[y''] + 3\mathcal{L}[y'] + 2\mathcal{L}[y] = \mathcal{L}[u_1]$$

$$s^2\mathcal{L}[y] - sy(0) - y'(0) + 3s\mathcal{L}[y] - 3y(0) + 2\mathcal{L}[y] = \frac{e^{-s}}{s}$$

$$(s^2 + 3s + 2)\mathcal{L}[y] - 1 = \frac{e^{-s}}{s}$$

$$\mathcal{L}[y] = \frac{e^{-s}}{s(s+1)(s+2)} + \frac{1}{(s+1)(s+2)}$$

Taking the inverse Laplace transform of both sides,

$$\begin{aligned} y &= \mathcal{L}^{-1}\left[\frac{e^{-s}}{s(s+1)(s+2)} + \frac{1}{(s+1)(s+2)}\right] \\ &= \mathcal{L}^{-1}\left[\frac{e^{-s}}{s(s+1)(s+2)}\right] + \mathcal{L}^{-1}\left[\frac{1}{(s+1)(s+2)}\right] \\ &= \mathcal{L}^{-1}\left[e^{-s}\frac{1}{s(s+1)(s+2)}\right] + \mathcal{L}^{-1}\left[\frac{1}{(s+1)(s+2)}\right] \end{aligned}$$

First we will find $\mathcal{L}^{-1}\left[e^{-s}\frac{1}{s(s+1)(s+2)}\right]$. We use that $\mathcal{L}^{-1}[e^{-cs}F(s)] = u_c(t)f(t-c)$ where $f(t) = \mathcal{L}^{-1}[F]$. In this notation, we have $c = 1$ and $F(s) = \frac{1}{s(s+1)(s+2)}$. So, we need to find $\mathcal{L}^{-1}\left[\frac{1}{s(s+1)(s+2)}\right]$. We use the method of partial fractions to rewrite F. The partial fractions decomposition is

$$\frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

Solving for the constants, we find $A = 1/2$, $B = -1$ and $C = 1/2$. Then

$$F(s) = \frac{1}{s(s+1)(s+2)} = \frac{1}{2s} - \frac{1}{s+1} + \frac{1}{2(s+2)}$$

Thus,

$$
\begin{aligned}
f(t) &= \mathcal{L}^{-1}[F] \\
&= \mathcal{L}^{-1}\left[\frac{1}{2s} - \frac{1}{s+1} + \frac{1}{2(s+2)}\right] \\
&= \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s}\right] - \mathcal{L}^{-1}\left[\frac{1}{s+1}\right] + \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s+2}\right] \\
&= \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}
\end{aligned}
$$

Then $\mathcal{L}^{-1}\left[e^{-s}\frac{1}{s(s+1)(s+2)}\right] = u_1(t)f(t-1) = u_1(t)\left[\frac{1}{2} - e^{-(t-1)} + \frac{1}{2}e^{-2(t-1)}\right]$.

Now we find $\mathcal{L}^{-1}\left[\frac{1}{(s+1)(s+2)}\right]$. Using partial fractions, we rewrite

$$\frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$$

Then

$$
\begin{aligned}
\mathcal{L}^{-1}\left[\frac{1}{(s+1)(s+2)}\right] &= \mathcal{L}^{-1}\left[\frac{1}{s+1} - \frac{1}{s+2}\right] \\
&= \mathcal{L}^{-1}\left[\frac{1}{s+1}\right] - \mathcal{L}^{-1}\left[\frac{1}{s+2}\right] \\
&= e^{-t} - e^{-2t}
\end{aligned}
$$

Putting these together,

$$
\begin{aligned}
y &= u_1(t)\left[\frac{1}{2} - e^{-(t-1)} + \frac{1}{2}e^{-2(t-1)}\right] + e^{-t} - e^{-2t} \\
&= \begin{cases} e^{-t} - e^{-2t} & t < 1 \\ \frac{1}{2} + (1-e)e^{-t} + \left(\frac{1}{2}e^2 - 1\right)e^{-2t} & t \geq 1 \end{cases}
\end{aligned}
$$

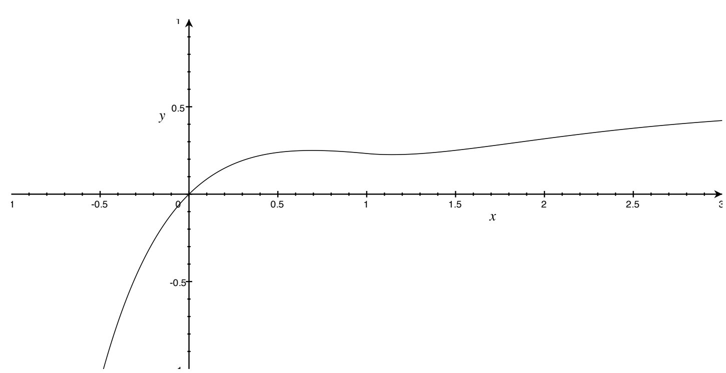

Figure 2

Problem 3. Find the solution of the inhomogeneous ODE

$$y'' + y = f(t)$$

with initial data $y(0) = y'(0) = 0$, where

$$f(t) = \begin{cases} t & 0 \le t < 2 \\ 2 & 2 \le t \end{cases}$$

First we rewrite f and a sum of Heaviside functions: $f(t) = t - u_2(t)(t-2)$. The equation is then

$$y'' + y = t - u_2(t)(t-2)$$

Taking the Laplace transform of both sides,

$$\begin{aligned} \mathcal{L}[y'' + y] &= \mathcal{L}[t - u_2(t)(t-2)] \\ \mathcal{L}[y''] + \mathcal{L}[y] &= \mathcal{L}[t] - \mathcal{L}[u_2(t)(t-2)] \\ s^2\mathcal{L}[y] - sy(0) - y'(0) + \mathcal{L}[y] &= \frac{1}{s^2} - \frac{e^{-2s}}{s^2} \\ (s^2+1)\mathcal{L}[y] &= \frac{1}{s^2} - \frac{e^{-2s}}{s^2} \\ \mathcal{L}[y] &= \frac{1}{s^2(s^2+1)} - \frac{e^{-2s}}{s^2(s^2+1)} \end{aligned}$$

Taking the inverse Laplace transform of both sides,

$$\begin{aligned} y &= \mathcal{L}^{-1}\left[\frac{1}{s^2(s^2+1)} - \frac{e^{-2s}}{s^2(s^2+1)}\right] \\ &= \mathcal{L}^{-1}\left[\frac{1}{s^2(s^2+1)}\right] - \mathcal{L}^{-1}\left[\frac{e^{-2s}}{s^2(s^2+1)}\right] \\ &= \mathcal{L}^{-1}\left[\frac{1}{s^2(s^2+1)}\right] - \mathcal{L}^{-1}\left[e^{-2s}\frac{1}{s^2(s^2+1)}\right] \end{aligned}$$

The partial fractions decomposition of $F(s) = \frac{1}{s^2(s^2+1)}$ is

$$\frac{1}{s^2(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+1} = \frac{1}{s^2} - \frac{1}{s^2+1}$$

Then

$$\begin{aligned} \mathcal{L}^{-1}\left[\frac{1}{s^2(s^2+1)}\right] &= \mathcal{L}^{-1}\left[\frac{1}{s^2} + -\frac{1}{s^2+1}\right] \\ &= \mathcal{L}^{-1}\left[\frac{1}{s^2}\right] - \mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] \\ &= t - \sin t \end{aligned}$$

Similarly, using $\mathcal{L}^{-1}[e^{-cs}F(s)] = u_c(t)f(t-c)$, we find $\mathcal{L}^{-1}\left[e^{-2s}\frac{1}{s^2(s^2+1)}\right] = u_2(t)[t-2-\sin(t-2)]$.

Putting these together,

$$\begin{aligned} y &= t - \sin t - u_2(t)[t-2-\sin(t-2)] \\ &= \begin{cases} t - \sin t & 0 \le t < 2 \\ 2 - \sin t + \sin(t-2) & t \ge 2 \end{cases} \end{aligned}$$

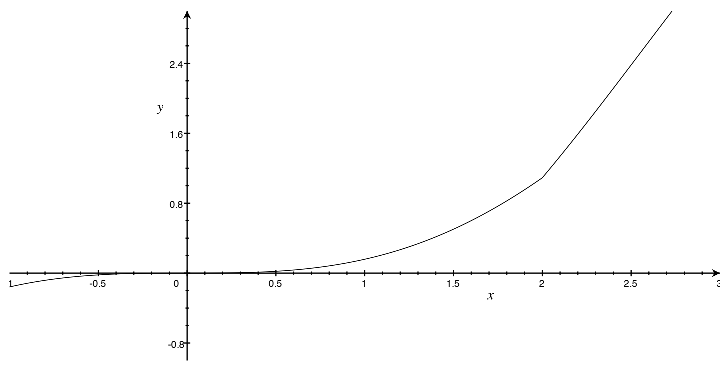

FIGURE 3

Problem 4. Find the solution of the inhomogeneous ODE

$$y'' - 4y = u_2(t) - u_1(t)$$

with initial conditions $y(0) = 0$ and $y'(0) = 1$.

Taking the Laplace transform of both sides,

$$\begin{aligned} \mathcal{L}[y'' - 4y] &= \mathcal{L}[u_2(t) - u_1(t)] \\ \mathcal{L}[y''] - 4\mathcal{L}[y] &= \mathcal{L}[u_2] - \mathcal{L}[u_1] \\ s^2\mathcal{L}[y] - sy(0) - y'(0) - 4\mathcal{L}[y] &= \frac{e^{-2s}}{s} - \frac{e^{-s}}{s} \\ (s^2-4)\mathcal{L}[y] - 1 &= \frac{e^{-2s}}{s} - \frac{e^{-s}}{s} \\ \mathcal{L}[y] &= \frac{1}{(s-2)(s+2)} + \frac{e^{-2s}}{s(s-2)(s+2)} - \frac{e^{-s}}{s(s-2)(s+2)} \end{aligned}$$

Taking the inverse Laplace transform of both sides,

$$
\begin{aligned}
y &= \mathcal{L}^{-1}\left[\frac{1}{(s-2)(s+2)} + \frac{e^{-2s}}{s(s-2)(s+2)} - \frac{e^{-s}}{s(s-2)(s+2)}\right] \\
&= \frac{1}{4}e^{2t} - \frac{1}{4}e^{-2t} - u_1(t)\left[-\frac{1}{4} + \frac{1}{8}e^{2(t-1)} + \frac{1}{8}e^{-2(t-1)}\right] + u_2(t)\left[-\frac{1}{4} + \frac{1}{8}e^{2(t-2)} + \frac{1}{8}e^{-2(t-2)}\right] \\
&= \begin{cases} \frac{1}{4}e^{2t} - \frac{1}{4}e^{-2t} & t < 1 \\ \frac{1}{4} + (\frac{1}{4} - \frac{1}{8}e^{-2})e^{2t} + (-\frac{1}{4} - \frac{1}{8}e^{2})e^{-2t} & 1 \le t < 2 \\ (\frac{1}{4} - \frac{1}{8}e^{-2} + \frac{1}{8}e^{-4})e^{2t} + (-\frac{1}{4} - \frac{1}{8}e^{2} + \frac{1}{8}e^{4})e^{-2t} & t \ge 2 \end{cases}
\end{aligned}
$$

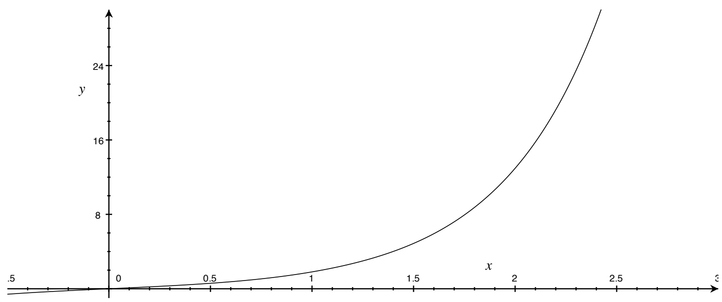

FIGURE 4