Homework 11, M 331
due 4/30/09

Problem 1. Find the solution of the inhomogeneous ODE

$$y'' + y = \delta(t - \pi)$$

with initial conditions $y(0) = y'(0) = 0$.

Taking the Laplace transform of both sides,

$$\begin{aligned} \mathcal{L}[y'' + y] &= \mathcal{L}[\delta(t-\pi)] \\ \mathcal{L}[y''] + \mathcal{L}[y] &= e^{-\pi s} \\ s^2\mathcal{L}[y] - sy(0) - y'(0) + \mathcal{L}[y] &= e^{-\pi s} \\ (s^2+1)\mathcal{L}[y] &= e^{-\pi s} \\ \mathcal{L}[y] &= \frac{e^{-\pi s}}{s^2+1} \end{aligned}$$

Taking the inverse Laplace transform of both sides,

$$\begin{aligned} y &= \mathcal{L}^{-1}\left[\frac{e^{-\pi s}}{s^2+1}\right] \\ &= u_\pi(t)\sin(t-\pi) \end{aligned}$$

Problem 2. Find the solution of the inhomogeneous ODE

$$y'' + 3y' + 2y = \delta(t - 5)$$

with initial conditions $y(0) = 0$ and $y'(0) = 1$.

Taking the Laplace transform of both sides,

$$\begin{aligned} \mathcal{L}[y'' + 3y' + 2y] &= \mathcal{L}[\delta(t-5)] \\ \mathcal{L}[y''] + 3\mathcal{L}[y'] + 2\mathcal{L}[y] &= e^{-5s} \\ s^2\mathcal{L}[y] - sy(0) - y'(0) + 3s\mathcal{L}[y] - 3y(0) + 2\mathcal{L}[y] &= e^{-5s} \\ (s^2+3s+2)\mathcal{L}[y] - 1 &= e^{-5s} \\ \mathcal{L}[y] &= \frac{e^{-5s}}{(s+1)(s+2)} + \frac{1}{(s+1)(s+2)} \end{aligned}$$

Taking the inverse Laplace transform of both sides,

$$\begin{aligned}
y &= \mathcal{L}^{-1}\left[\frac{e^{-5s}}{(s+1)(s+2)} + \frac{1}{(s+1)(s+2)}\right] \\
&= \mathcal{L}^{-1}\left[\frac{e^{-5s}}{(s+1)(s+2)}\right] + \mathcal{L}^{-1}\left[\frac{1}{(s+1)(s+2)}\right] \\
&= \mathcal{L}^{-1}\left[e^{-5s}\frac{1}{(s+1)(s+2)}\right] + \mathcal{L}^{-1}\left[\frac{1}{(s+1)(s+2)}\right]
\end{aligned}$$

First we will find $\mathcal{L}^{-1}\left[e^{-5s}\frac{1}{(s+1)(s+2)}\right]$. We use that $\mathcal{L}^{-1}[e^{-cs}F(s)] = u_c(t)f(t-c)$ where $f(t) = \mathcal{L}^{-1}[F]$. In this notation, we have $c = 5$ and $F(s) = \frac{1}{(s+1)(s+2)}$. So, we need to find $\mathcal{L}^{-1}\left[\frac{1}{(s+1)(s+2)}\right]$. We use the method of partial fractions to rewrite F. The partial fractions decomposition is

$$\frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

Solving for the constants, we find $A = 1$ and $B = -1$. Then

$$F(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$$

Thus,

$$\begin{aligned}
f(t) &= \mathcal{L}^{-1}[F] \\
&= \mathcal{L}^{-1}\left[\frac{1}{s+1} - \frac{1}{s+2}\right] \\
&= \mathcal{L}^{-1}\left[\frac{1}{s+1}\right] - \mathcal{L}^{-1}\left[\frac{1}{s+2}\right] \\
&= e^{-t} - e^{-2t}
\end{aligned}$$

Then $\mathcal{L}^{-1}\left[e^{-5s}\frac{1}{(s+1)(s+2)}\right] = u_5(t)f(t-5) = u_5(t)[e^{-(t-5)} - e^{-2(t-5)}]$.

And we have also computed the second term, $\mathcal{L}^{-1}\left[\frac{1}{(s+1)(s+2)}\right] = e^{-t} - e^{-2t}$.

Putting these together,

$$y = e^{-t} - e^{-2t} + u_5(t)[e^{-(t-5)} - e^{-2(t-5)}]$$

Problem 3. Consider the matrix ODE

$$y' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} y$$

(i) Find the general solution. Where does the solution tend to as $t \to \infty$?

Let $A = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$. First we find the eigenvalues of A.

$$\begin{aligned} \det(\lambda I - A) &= \det \begin{pmatrix} \lambda - 3 & 2 \\ -2 & \lambda + 2 \end{pmatrix} \\ &= (\lambda - 3)(\lambda + 2) - (2)(-2) \\ &= \lambda^2 - \lambda - 2 \\ &= (\lambda - 2)(\lambda + 1) \end{aligned}$$

Setting this equal to zero, we find the eigenvalues of A are $\lambda_1 = 2$ and $\lambda_2 = -1$. Now we find eigenvectors corresponding to these eigenvalues.

Suppose $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ is an eigenvector with eigenvalue $\lambda_1 = 2$. Then $Av = 2v$ gives

$$\begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 2 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\begin{pmatrix} 3v_1 - 2v_2 \\ 2v_1 - 2v_2 \end{pmatrix} = \begin{pmatrix} 2v_1 \\ 2v_2 \end{pmatrix}$$

$$\begin{aligned} 3v_1 - 2v_2 &= 2v_1 \\ 2v_1 - 2v_2 &= 2v_2 \end{aligned}$$

or

$$\begin{aligned} v_1 - 2v_2 &= 0 \\ 2v_1 - 4v_2 &= 0 \end{aligned}$$

Thus, both equations give $v_2 = \frac{1}{2} v_1$. So, we can take $v_1 = 2$ which implies $v_2 = 1$ and we get the eigenvector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Then one solution of the equation is $u_1 = e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Now suppose $w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ is an eigenvector with eigenvalue $\lambda_2 = -1$. Then $Aw = -w$ gives

$$\begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = - \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

$$\begin{pmatrix} 3w_1 - 2w_2 \\ 2w_1 - 2w_2 \end{pmatrix} = \begin{pmatrix} -w_1 \\ -w_2 \end{pmatrix}$$

$$3w_1 - 2w_2 = -w_1$$
$$2w_1 - 2w_2 = -w_2$$

or

$$4w_1 - 2w_2 = 0$$
$$2w_1 - w_2 = 0$$

Thus, both equations give $w_2 = 2w_1$. So, we can take $w_1 = 1$ which implies $w_2 = 2$ and we get the eigenvector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Then another solution of the equation is $u_2 = e^{-t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

So, the general solution is $y = c_1 e^{2t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. As $t \to \infty$, $e^{-t} \to 0$ for any constant c_2. Thus, if $c_1 = 0$, the solution tends toward the equilibrium point $(0, 0)$ as $t \to \infty$. For all other solutions y, $|y| \to \infty$ as $t \to \infty$ in the direction of the eigenspace for $\lambda_1 = 2$, $y_2 = \frac{1}{2} y_1$.

(ii) Find the solution with initial condition $y(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

We substitute $t = 0$, $y_1 = 1$ and $y_2 = 1$ into the general solution $y = c_1 e^{2t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. This gives

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

or

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2c_1 + c_2 \\ c_1 + 2c_2 \end{pmatrix}$$

which gives the system of equations

$$2c_1 + c_2 = 1$$
$$c_1 + 2c_2 = 1$$

Solving this system, we find $c_1 = c_2 = \frac{1}{3}$. So, the solution to the IVP is $y = \frac{1}{3} e^{2t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \frac{1}{3} e^{-t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

(iii) Draw eigen directions and sketch some trajectories, in particular the one with the previous initial condition.

Figure 1 shows the eigenspaces and the solution to the IVP. All other trajectories follow the vector field.

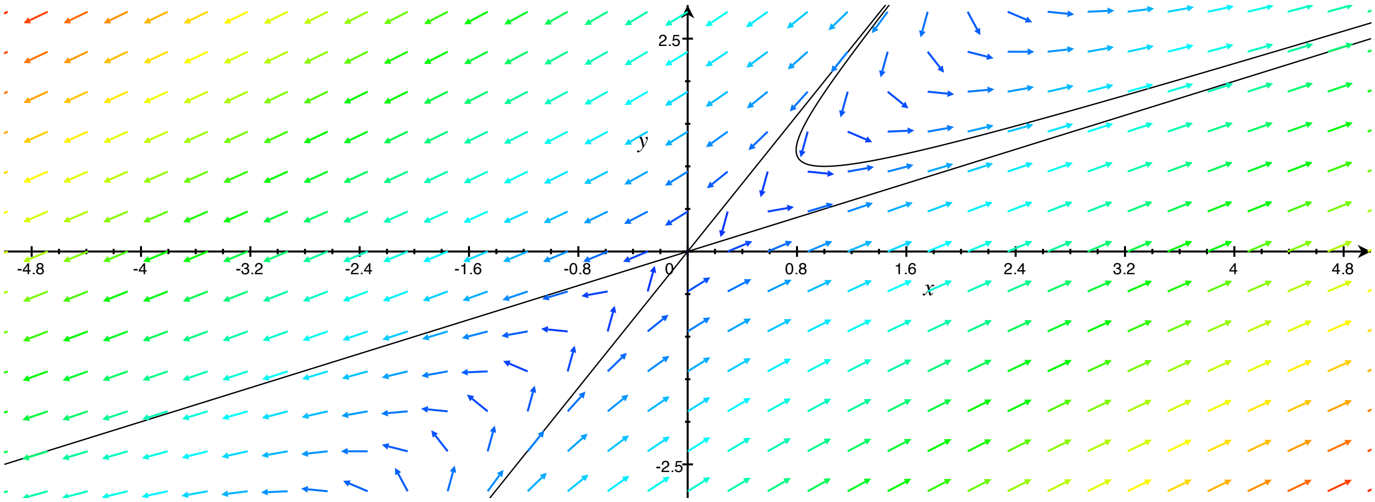

FIGURE 1

Problem 4. Find the solution to the matrix ODE

$$y' = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} y$$

which satisfies $y(0) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$. Draw eigen directions and sketch the solution you found.

The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 3$ and eigenvectors are $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, respectively. So, the general solution is $y = c_1 e^t \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. The solution to the IVP is $y = e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Below are the eigenspaces and the solution to the IVP.

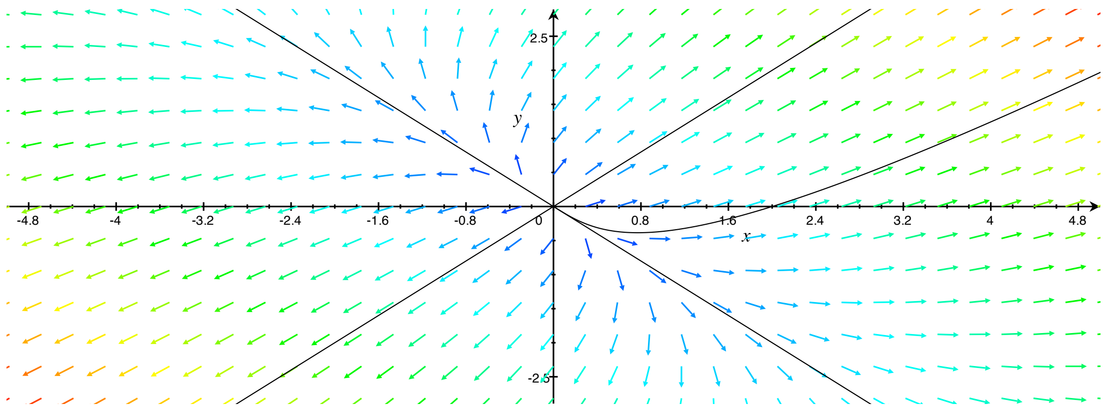

FIGURE 2