PRACTICE FINAL, M 331, FOR FINAL EXAM ON TUESDAY 5/19/09, 1:30 PM

You can use any method to solve the ODEs below (except computer software such as mathematica etc). Show all work.

Problem 1. Find the general solution to the ODE

$$y'' - 2y' + y = \frac{e^t}{t}$$

First we find y_h by solving the homogeneous ODE $y_h'' - 2y_h' + y_h = 0$

The characteristic equation is $\lambda^2 - 2\lambda + 1 = 0$. Factoring,

$$\begin{aligned} \lambda^2 - 2\lambda + 1 &= 0 \\ (\lambda - 1)(\lambda - 1) &= 0 \\ \lambda &= 1 \end{aligned}$$

Then the fundamental solutions are $y_1 = e^t$ and $y_2 = te^t$.

We use variation of parameters to find a particular solution. We look for a solution of the form $y_p = c_1(t)y_1(t) + c_2(t)y_2(t)$ where $c_1(t)$ and $c_2(t)$ are unknown functions of t and y_1 and y_2 are the fundamental solutions of the homogeneous equation. We have the following formulas for c_1 and c_2:

$$\begin{aligned} c_1(t) &= -\int \frac{y_2 f}{W} \mathrm{d}t \\ c_2(t) &= \int \frac{y_1 f}{W} \mathrm{d}t \end{aligned}$$

where $W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$ is the Wronskian determinant.

We compute

$$\begin{aligned} y_1 &= e^t \\ y_1' &= e^t \end{aligned}$$

$$\begin{aligned} y_2 &= te^t \\ y_2' &= (1+t)e^t \end{aligned}$$

$$W(t) = y_1 y_2' - y_1' y_2 = (1+t)e^{2t} - te^{2t} = e^{2t}$$

Now we use our formulas to find c_1 and c_2.

$$\begin{aligned} c_1(t) &= -\int \frac{y_2 f}{W} \mathrm{d}t \\ &= -\int \frac{te^t(\frac{e^t}{t})}{e^{2t}} \mathrm{d}t \\ &= -\int 1 \mathrm{d}t \\ &= -t \end{aligned}$$

$$\begin{aligned} c_2(t) &= \int \frac{y_1 f}{W} \mathrm{d}t \\ &= \int \frac{e^t(\frac{e^t}{t})}{e^{2t}}) \mathrm{d}t \\ &= \int \frac{1}{t} \mathrm{d}t \\ &= \ln t \end{aligned}$$

Then $y_p = -te^t + te^t \ln t$ is one solution. Since $-te^t$ is a solution of the homogeneous ODE, we can omit this term and use $y_p = te^t \ln t$. Therefore, the general solution is $y = te^t \ln t + c_1 e^t + c_2 te^t$.

Problem 2. Consider the ODE

$$y' = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix} y$$

(i) Find the solution with initial condition $y(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Let $A = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix}$. First we find the eigenvalues of A.

$$\begin{aligned} \det(\lambda I - A) &= \det \begin{pmatrix} \lambda - 1 & -1 \\ 4 & \lambda - 1 \end{pmatrix} \\ &= \lambda^2 - 2\lambda + 5 \end{aligned}$$

Setting this equal to zero, we find the eigenvalues of A are $1 \pm 2i$. Let $\lambda_1 = 1 + 2i$ and $\lambda_2 = 1 - 2i$. Now we find eigenvectors corresponding to these eigenvalues.

Suppose $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ is an eigenvector with eigenvalue $\lambda_1 = 1 + 2i$. Then $Av = (1 + 2i)v$ gives

$$\begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = (1 + 2i) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\begin{pmatrix} v_1 + v_2 \\ -4v_1 + v_2 \end{pmatrix} = \begin{pmatrix} (1 + 2i)v_1 \\ (1 + 2i)v_2 \end{pmatrix}$$

$$v_1 + v_2 = (1 + 2i)v_1$$

$$-4v_1 + v_2 = (1 + 2i)v_2$$

or

$$v_2 = 2iv_1$$

$$-4v_1 = 2iv_2$$

Putting $v_1 = 1$ in the first equation, we get $v_2 = 2i$. So, an eigenvector for $\lambda_1 = 1 + 2i$ is $v = \begin{pmatrix} 1 \\ 2i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix} i$. Consequently, an eigenvector for the conjugate eigenvalue $\lambda_2 = 1 - 2i$ is the conjugate vector $w = \begin{pmatrix} 1 \\ -2i \end{pmatrix}$.

Allowing complex solutions, the general solution is $y = c_1 e^{(1+2i)t} \begin{pmatrix} 1 \\ 2i \end{pmatrix} + c_2 e^{(1-2i)t} \begin{pmatrix} 1 \\ -2i \end{pmatrix}$.

Using Euler's formula and the superposition principle, in terms of real functions, the general solution is $y = c_1 e^t \cos(2t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} - c_1 e^t \sin(2t) \begin{pmatrix} 0 \\ 2 \end{pmatrix} + c_2 e^t \sin(2t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^t \cos(2t) \begin{pmatrix} 0 \\ 2 \end{pmatrix}$.

Substituting $t = 0$, $y_1 = 0$ and $y_2 = 1$ into the general solution gives

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

or

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 \\ 2c_2 \end{pmatrix}$$

Thus, $c_1 = 0$ and $c_2 = 1/2$. So, the solution to the IVP is $y = \frac{1}{2} e^t \sin(2t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{2} e^t \cos(2t) \begin{pmatrix} 0 \\ 2 \end{pmatrix}$.

(ii) Draw a picture of the general behavior of the solutions and characterize whether the origin is a source, sink, saddle or spiral point.

Figure 1 shows the vector field and the solution to the IVP. Since the real parts of the eigenvalues are positive, we have an unstable spiral.

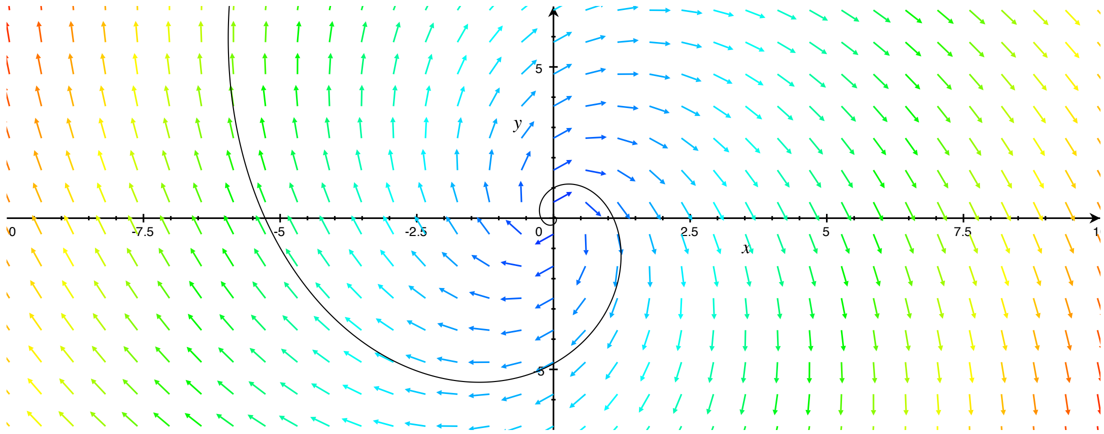

FIGURE 1

Problem 3. Consider the matrix ODE

$$y' = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} y$$

(i) Find the general solution.

Let $A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$. First we find the eigenvalues of A.

$$\begin{aligned} \det(\lambda I - A) &= \det \begin{pmatrix} \lambda - 1 & -3 \\ -3 & \lambda - 1 \end{pmatrix} \\ &= (\lambda - 1)(\lambda - 1) - (-3)(-3) \\ &= \lambda^2 - 2\lambda - 8 \\ &= (\lambda - 4)(\lambda + 2) \end{aligned}$$

Setting this equal to zero, we find the eigenvalues of A are $\lambda_1 = 4$ and $\lambda_2 = -2$. Now we find eigenvectors corresponding to these eigenvalues.

Suppose $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ is an eigenvector with eigenvalue $\lambda_1 = 4$. Then $Av = 4v$ gives

$$\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 4 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\begin{pmatrix} v_1 + 3v_2 \\ 3v_1 + v_2 \end{pmatrix} = \begin{pmatrix} 4v_1 \\ 4v_2 \end{pmatrix}$$

$$\begin{aligned} v_1 + 3v_2 &= 4v_1 \\ 3v_1 + v_2 &= 4v_2 \end{aligned}$$

or

$$-3v_1 + 3v_2 = 0$$

$$3v_1 - 3v_2 = 0$$

Thus, both equations give $v_2 = v_1$. So, we can take $v_1 = 1$ which implies $v_2 = 1$ and we get the eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Then one solution of the equation is $u_1 = e^{4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Now suppose $w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ is an eigenvector with eigenvalue $\lambda_2 = -2$. Then $Aw = -2w$ gives

$$\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = -2 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

$$\begin{pmatrix} w_1 + 3w_2 \\ 3w_1 + w_2 \end{pmatrix} = \begin{pmatrix} -2w_1 \\ -2w_2 \end{pmatrix}$$

$$w_1 + 3w_2 = -2w_1$$

$$3w_1 + w_2 = -2w_2$$

or

$$3w_1 + 3w_2 = 0$$

$$3w_1 + 3w_2 = 0$$

Thus, both equations give $w_2 = -w_1$. So, we can take $w_1 = 1$ which implies $w_2 = -1$ and we get the eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Then another solution of the equation is $u_2 = e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

So, the general solution is $y = c_1 e^{4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

(ii) Sketch the general behavior of the solutions and characterize whether the origin is a source, sink, saddle or spiral point.

Figure 2 shows the eigensolutions and the vector field $f(y_1, y_2) = \begin{pmatrix} y_1 + 3y_2 \\ 3y_1 + y_2 \end{pmatrix}$. The origin is a saddle point.

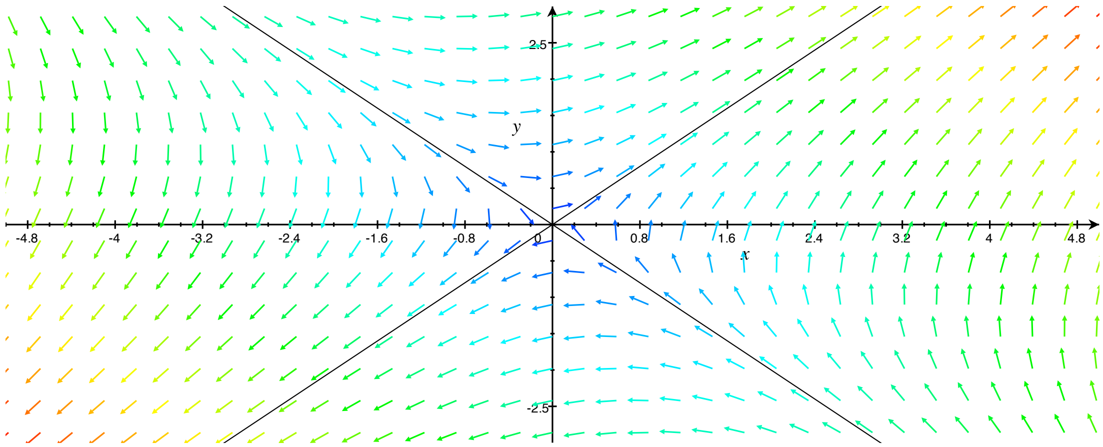

FIGURE 2

Problem 4. Solve the linear ODE

$$y'' + 9y = \delta(t-1)$$

with initial conditions $y(0) = 0$ and $y'(0) = 1$.

Taking the Laplace transform of both sides,

$$\begin{gathered}
\mathcal{L}[y'' + 9y] = \mathcal{L}[\delta_1] \\
\mathcal{L}[y''] + 9\mathcal{L}[y] = e^{-s} \\
s^2\mathcal{L}[y] - sy(0) - y'(0) + 9\mathcal{L}[y] = e^{-s} \\
(s^2+9)\mathcal{L}[y] - 1 = e^{-s} \\
\mathcal{L}[y] = \frac{e^{-s}}{s^2+9} + \frac{1}{s^2+9}
\end{gathered}$$

Taking the inverse Laplace transform of both sides,

$$\begin{aligned}
y &= \mathcal{L}^{-1}\left[\frac{e^{-s}}{s^2+9} + \frac{1}{s^2+9}\right] \\
&= \mathcal{L}^{-1}\left[\frac{e^{-s}}{s^2+9}\right] + \mathcal{L}^{-1}\left[\frac{1}{s^2+9}\right] \\
&= \frac{1}{3}\mathcal{L}^{-1}\left[e^{-s}\frac{3}{s^2+9}\right] + \frac{1}{3}\mathcal{L}^{-1}\left[\frac{3}{s^2+9}\right] \\
&= \frac{1}{3}\mathcal{L}^{-1}\left[e^{-s}\frac{3}{s^2+9}\right] + \frac{1}{3}\sin(3t)
\end{aligned}$$

To find $\mathcal{L}^{-1}\left[e^{-s}\frac{3}{s(s^2+9)}\right]$ we use that $\mathcal{L}^{-1}[e^{-cs}F(s)] = u_c(t)f(t-c)$ where $f(t) = \mathcal{L}^{-1}[F]$. In this notation, we have $c = 1$ and $F(s) = \frac{3}{s^2+9}$. Since $\mathcal{L}^{-1}\left[\frac{3}{s^2+9}\right] = \sin(3t)$, we have $f(t) = \sin(3t)$.

Then $f(t-1) = \sin(3(t-1))$. So, $\mathcal{L}^{-1}\left[e^{-s}\frac{3}{s(s^2+9)}\right] = u_1(t)\sin(3(t-1))$. Putting this into the above,

$$\begin{aligned} y &= \frac{1}{3}\mathcal{L}^{-1}\left[e^{-s}\frac{3}{s^2+9}\right] + \frac{1}{3}\sin(3t) \\ &= \frac{1}{3}\sin(3t) + \frac{1}{3}u_1(t)\sin(3(t-1)) \end{aligned}$$