HOMEWORK 1, M 331
DUE 2/12/09

Problem 1. Given the functions $g_1(x) = \ln(x)$, $g_2(x) = e^{1-x}$, $g_3(x) = 1/x$, and $g_4(x) = x\sin(\pi x^2)$, find in each case

(i) a function $f_i(x)$ whose derivative is $g_i(x)$.

Let

$$f_1(x) = x\ln x - x$$
$$f_2(x) = -e^{1-x}$$
$$f_3(x) = \ln x$$
$$f_4(x) = -\frac{1}{2\pi}\cos(\pi x^2)$$

(ii) Is there another function $h_i(x)$ different from $f_i(x)$ whose derivative is $g_i(x)$? If so, how many functions are there whose derivative is $g_i(x)$?

If $h_i(x) = f_i(x) + C$ for any constant C, then the derivative of $h_i(x)$ is also $g_i(x)$. Thus, there are infinitely many such functions.

(iii) Among all the functions $f_i(x)$ whose derivative is $g_i(x)$, find the one which satisfies $f_i(1) = -1$.

$$f_1(x) = x\ln x - x$$
$$f_2(x) = -e^{1-x}$$
$$f_3(x) = \ln x - 1$$
$$f_4(x) = -\frac{1}{2\pi}\cos(\pi x^2) - \frac{1}{2\pi} - 1$$

Problem 2. The population of the US was 8.6 million in 1820 and 40 million in 1897.

(i) Write down the differential equation for population growth in this case.

We use an exponential growth model. Let $P = P(t)$ be the population of the US in millions t years after 1820. The differential equation for P is $P' = rP$, where r is the growth rate. The solution is $P(t) = P_0e^{rt}$, where $P_0 = P(0)$ is the initial value. Since $P(0) = 8.6$, we have $P(t) = 8.6e^{rt}$. To find r we use the information $P(77) = 40$.

$$40 = 8.6e^{77r}$$
$$40/8.6 = e^{77r}$$
$$\ln(40/8.6) = \ln(e^{77r})$$
$$\ln(40/8.6) = 77r$$
$$r = \frac{\ln(40/8.6)}{77} \approx 0.02$$

So, $P(t) = 8.6e^{0.02t}$.

(ii) Calculate the number of people in the US according to your model in the year 2003. How well does this growth model work: the actual population of the US in 2003 was 291 million?

$P(183) = 8.6e^{0.02(183)} \approx 332$

(iii) How long does it take for the population of the US to double according to your growth model?

We wish to find t when $P(t) = 2P_0 = 17.2$.

$$\begin{aligned} 17.2 &= 8.6e^{0.02t} \\ 2 &= e^{0.02t} \\ \ln 2 &= \ln(e^{0.02t}) \\ 0.02t &= \ln 2 \\ t &= (\ln 2)/0.02 \approx 35 \end{aligned}$$

Therefore, according to this model, the population of the US doubled by 1855.

Problem 3. Calculate the impact velocity if you jump off a 7 foot wall (ignore air drag).

Ignoring air drag, the differential equation for velocity is $\frac{\mathrm{d}v}{\mathrm{d}t} = 32.2$, where $g = 32.2\text{ft/s}^2$ is the acceleration due to gravity. Integrating,

$$\begin{aligned} \int \frac{\mathrm{d}v}{\mathrm{d}t}\mathrm{d}t &= \int 32.2\mathrm{d}t \\ v(t) &= 32.2t + C \end{aligned}$$

Substituting $t = 0$, we see that $C = v(0)$ is our initial velocity. Thus, $C = 0$ and the equation for velocity is $v(t) = 32.2t$.

To compute the impact velocity, we need to know the time of impact. So, we find the equation for displacement, s. The differential equation is $\frac{\mathrm{d}s}{\mathrm{d}t} = v(t) = 32.2t$. Integrating,

$$\begin{aligned} \int \frac{\mathrm{d}s}{\mathrm{d}t}\mathrm{d}t &= \int 32.2t\mathrm{d}t \\ s(t) &= 16.6t^2 + C \end{aligned}$$

Substituting $t = 0$, we see that $C = s(0)$ is our initial displacement. Since we measure from our initial position, atop the wall and take downwards as the positive direction, $C = 0$ and the equation for displacement is $s(t) = 16.6t^2$. Thus, we need to find t when we hit the ground, i.e. $s = 7$.

$$\begin{aligned} 7 &= 16.6t^2 \\ t &= \sqrt{7/16.6} \end{aligned}$$

Putting this value back into the equation for velocity, the impact velocity is $v = 32.2\sqrt{7/16.6} \approx 20.9\text{ft/s}$.

Problem 4. Omit this problem.

Problem 5. What is the minimal height above ground so that an object of mass 10 kg dropped from this height hits the ground at a speed within 5% of its terminal velocity, assuming the air drag coefficient is 2 kg/sec?

We need to find the equations for velocity, v and displacement, s.

The differential equation for velocity is $\frac{\mathrm{d}v}{\mathrm{d}t} = 9.8 - \frac{\gamma}{m}v$, where $g = 9.8\mathrm{m/s}^2$ is the acceleration due to gravity, m is the mass of the object, and γ is the drag coefficient. Separating variables,

$$\int \frac{1}{9.8 - \frac{\gamma}{m}v}\mathrm{d}v = \int 1\mathrm{d}t$$
$$-\frac{m}{\gamma}\ln(9.8 - \frac{\gamma}{m}v) = t + C$$
$$\ln(9.8 - \frac{\gamma}{m}v) = -\frac{\gamma}{m}t + C$$
$$e^{\ln(9.8-\frac{\gamma}{m}v)} = e^{-\frac{\gamma}{m}t+C}$$
$$9.8 - \frac{\gamma}{m}v = e^{-\frac{\gamma}{m}t}e^C$$
$$-\frac{\gamma}{m}v = Ce^{-\frac{\gamma}{m}t} - 9.8$$
$$v(t) = Ce^{-\frac{\gamma}{m}t} + \frac{9.8m}{\gamma}$$

Setting $t = 0$, we have

$$v(0) = C + \frac{9.8m}{\gamma}$$

So, $C = v_0 - \frac{9.8m}{\gamma}$. The equation for velocity is then

$$v(t) = (v_0 - \frac{9.8m}{\gamma})e^{-\frac{\gamma}{m}t} + \frac{9.8m}{\gamma}$$

Now we use that $v_0 = 0$, $m = 10$, and $\gamma = 2$. The equation becomes

$$v(t) = -49e^{-0.2t} + 49$$

Letting $t \to \infty$ in the above, the terminal velocity, v_t is 49 m/s. So, to be within 5% of the terminal velocity is to be in the interval (46.55, 51.45).

Now we need to find the equation for displacement. The differential equation is $\frac{\mathrm{d}s}{\mathrm{d}t} = v(t) = -49e^{-0.2t} + 49$. Integrating,

$$\int \frac{\mathrm{d}s}{\mathrm{d}t}\mathrm{d}t = \int(-49e^{-0.2t} + 49)\mathrm{d}t$$
$$s(t) = 245e^{-0.2t} + 49t + C$$

To find C, we use that $s(0) = 0$. Substituting $t = 0$ and $s = 0$, we get $C = -245$. So, our equation is

$$s(t) = 245e^{-0.2t} + 49t - 245$$

Now we wish to minimize the height so that our impact velocity is between 46.55 and 51.45. Noting that s is increasing for $t \geq 0$, minimizing the height is equivalent to finding the first time when our velocity falls in this range. But since v is also increasing, the time we want is when v first enters the interval, i.e. when $v = 46.55$. Substituting into the equation,

$$\begin{aligned}
46.55 &= -49e^{-0.2t} + 49 \\
-2.45 &= -49e^{-0.2t} \\
0.05 &= e^{-0.2t} \\
\ln(0.05) &= \ln(e^{-0.2t}) \\
\ln(0.05) &= -0.2t \\
t &= -5\ln(0.05) \approx 14.98
\end{aligned}$$

Putting this value into our equation for displacement,

$$s = 245e^{\ln(0.05)} + 49(-5\ln(0.05)) - 245 \approx 501$$

Therefore, the minimal height is about 501 meters above ground.

Problem 6. The population dynamics of rabbits in a certain habitat is described by

$$\frac{\mathrm{d}P}{\mathrm{d}t} = P - 500$$

where $P(t)$ denotes the rabbit population at time t (measured in months), and 500 is the amount of rabbits eaten by preying animals, etc. per month.

(i) For which initial rabbit population P_0 does the rabbit population stay constant over time?

For $P_0 = 500$, $\frac{\mathrm{d}P}{\mathrm{d}t} = 0$, and hence the population is constant.

(ii) For which initial rabbit populations do the rabbits die out?

If $0 < P_0 < 500$, then $\frac{\mathrm{d}P}{\mathrm{d}t} < 0$. Furthermore, differentiating the equation $\frac{\mathrm{d}P}{\mathrm{d}t} = P - 500$ with respect to t, we see that $\frac{\mathrm{d}^2P}{\mathrm{d}t^2} = \frac{\mathrm{d}P}{\mathrm{d}t} < 0$. So, P is decreasing and concave down. Hence, the population will eventually reach 0, i.e. will die out.

(iii) Can it also happen that the rabbits just keep growing despite the fact that some are eaten all the time? What initial population is needed for that to happen?

If $P_0 > 500$, then $\frac{\mathrm{d}P}{\mathrm{d}t} > 0$ and so the population will increase for all time.

Problem 7. Fish growth in a certain area of the ocean can be modeled by the logistic differential equation. Assume that $r = 0.7$ and the carrying capacity $K = 80 * 10^6$ kilograms (the mass of all the fish). If the initial population is $P_0 = 0.25K$ find the population (measured in kilograms) 2 years later. Also find the time when the population reaches 3/4 of the carrying capacity.

We use the logistic equation

$$P(t) = \frac{KP_0e^{rt}}{K + P_0(e^{rt} - 1)}$$

where P is the population measured in millions of kilograms (for simplicity). Then $r = 0.7$, $K = 80$ and $P_0 = 0.25K = 20$. The equation becomes

$$\begin{aligned} P(t) &= \frac{80 * 20e^{0.7t}}{80 + 20(e^{0.7t} - 1)} \\ &= \frac{1600e^{0.7t}}{80 + 20e^{0.7t} - 20} \\ &= \frac{80e^{0.7t}}{3 + e^{0.7t}} \end{aligned}$$

The population after two years is $P(2)$. Substituting into the equation, we have

$$P(2) = \frac{80e^{0.7*2}}{3 + e^{0.7*2}} \approx 46$$

Remembering that we are working in millions of kilograms, after two years, the population is $46*10^6$ kilograms.

The time when the population reaches 3/4 of the carrying capacity is the time t when $P(t) = 0.75K = 60$. So we solve for t in the equation

$$60 = \frac{80e^{0.7t}}{3 + e^{0.7t}}$$

Multiplying both sides by $3 + e^{0.7t}$ gives

$$\begin{aligned} 60(3 + e^{0.7t}) &= 80e^{0.7t} \\ 180 + 60e^{0.7t} &= 80e^{0.7t} \\ 180 &= 20e^{0.7t} \\ 9 &= e^{0.7t} \end{aligned}$$

Now to get the t out of the exponent, we take the natural log of both sides and use the property that $\ln e^x = x$. We have

$$\ln 9 = \ln e^{0.7t}$$

$$\ln 9 = 0.7t$$
$$t = (\ln 9)/0.7 \approx 3.1$$

So, the population reaches the desired level in about 3 years.