Homework 3, M 331
due 2/26/09

Problem 1. Consider the ODE

$$(2ty^2 + 2y)\mathrm{d}t + (2t^2y + 2t)\mathrm{d}y = 0$$

(i) Verify that this equation is exact, i.e., of the form $M\mathrm{d}t + N\mathrm{d}y = 0$ with $M_y = N_t$ (where partial derivatives w.r.t. the corresponding variable are denoted by a subscript: $M_y = \frac{\partial M}{\partial y}$, etc.).

$$\begin{aligned} M(t,y) &= 2ty^2 + 2y \quad \text{so} \qquad M_y = 4ty + 2 \\ N(t,y) &= 2t^2y + 2t \quad \text{so} \qquad N_t = 4ty + 2 \end{aligned}$$

Therefore, the equation is exact.

(ii) Find the function $F(t,y)$ describing the solutions implicitly via $F(t,y) = C$.

The function F satisfies $\frac{\partial F}{\partial t} = M$ and $\frac{\partial F}{\partial y} = N$.

Integrating M with respect to t,

$$\int \frac{\partial F}{\partial t}\mathrm{d}t = \int (2ty^2 + 2y)\mathrm{d}t$$

(1) $$F(t,y) = t^2y^2 + 2ty + f(y)$$

where $f(y)$ is an unknown function of y.

Integrating N with respect to y,

$$\int \frac{\partial F}{\partial y}\mathrm{d}y = \int (2t^2y + 2t)\mathrm{d}y$$

(2) $$F(t,y) = t^2y^2 + 2ty + g(t)$$

where $g(t)$ is an unknown function of t.

Equating (1) and (2), we see that $F(t,y) = t^2y^2 + 2ty$.

(iii) Determine the constant C so that the solution fulfills $y(0) = 0$. Draw this solution somehow.

All solutions satisfy $t^2y^2 + 2ty = C$ for some constant C. Since $y = 0$ when $t = 0$, we have $C = 0$. Then the solution satisfies $t^2y^2 + 2ty = 0$. Factoring, $y(t^2y + 2t) = 0$ implies that $y = 0$ or $y = -\frac{2}{t}, t \neq 0$. Since our solution must satisfy $y(0) = 0$, it must be the equilibrium solution $y(t) = 0$ for all t.

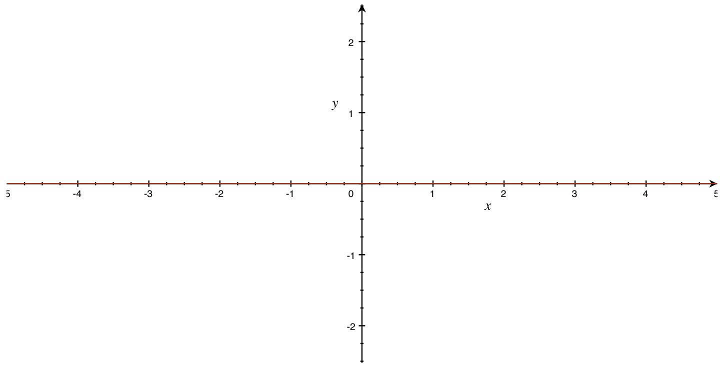

FIGURE 1

Problem 2. Consider the ODE

$$(ye^{2ty} + t)\mathrm{d}t + bte^{2ty}\mathrm{d}y = 0$$

(i) For which value of b is this ODE exact?

For the ODE to be exact, we need $M_y = N_t$. $M(t, y) = ye^{2ty} + t$ so $M_y = e^{2ty} + 2tye^{2ty}$. $N(t, y) = bte^{2ty}$ so $N_t = be^{2ty} + 2btye^{2ty}$. Setting $M_y = N_t$, we have

$$e^{2ty} + 2tye^{2ty} = be^{2ty} + 2btye^{2ty}$$

Factoring,

$$e^{2ty}(1 + 2ty) = be^{2ty}(1 + 2ty)$$

Therefore, $b = 1$.

(ii) Find all solutions of the ODE (possibly implicitly) for this value of b.

The ODE is

$$(ye^{2ty} + t)\mathrm{d}t + te^{2ty}\mathrm{d}y = 0$$

Since the equation is exact, there is a function $F = F(t, y)$ such that all solutions lie on the level curves of F, i.e. they satisfy $F(t, y) = C$ for some constant C. Furthermore, the function F satisfies $\frac{\partial F}{\partial t} = M$ and $\frac{\partial F}{\partial y} = N$.

Integrating M with respect to t,

$$\int \frac{\partial F}{\partial t}\mathrm{d}t = \int (ye^{2ty} + t)\mathrm{d}t$$

$$F(t, y) = \frac{1}{2}e^{2ty} + \frac{1}{2}t^2 + f(y) \tag{3}$$

where $f(y)$ is an unknown function of y.

Integrating N with respect to y,

$$\int \frac{\partial F}{\partial y}\mathrm{d}y = \int te^{2ty}\mathrm{d}y$$

$$F(t,y) = \frac{1}{2}e^{2ty} + g(t) \tag{4}$$

where $g(t)$ is an unknown function of t.

Equating (3) and (4), we see that $F(t,y) = \frac{1}{2}e^{2ty} + \frac{1}{2}t^2$. Therefore, all solutions of the ODE are $\frac{1}{2}e^{2ty} + \frac{1}{2}t^2 = C$ for some constant C.

Problem 3. Solve (possibly implicitly) the ODE $(2t-y)\mathrm{d}t + (2y-t)\mathrm{d}y = 0$ with initial condition $y(1) = 3$. Draw the solution curve.

The equation is in the form $M\mathrm{d}t + N\mathrm{d}y = 0$ with $M(t,y) = 2t - y$ and $N(t,y) = 2y - t$. Since $M_y = -1 = N_t$, the equation is exact. So, there is a function $F = F(t,y)$ such that $\frac{\partial F}{\partial t} = M$ and $\frac{\partial F}{\partial y} = N$ and all solutions lie on the level curves of F.

Integrating M with respect to t,

$$\int \frac{\partial F}{\partial t}\mathrm{d}t = \int (2t-y)\mathrm{d}t$$

$$F(t,y) = t^2 - ty + f(y) \tag{5}$$

where $f(y)$ is an unknown function of y.

Integrating N with respect to y,

$$\int \frac{\partial F}{\partial y}\mathrm{d}y = \int (2y-t)\mathrm{d}y$$

$$F(t,y) = y^2 - ty + g(t) \tag{6}$$

where $g(t)$ is an unknown function of t.

Equating (5) and (6), we see that $F(t,y) = t^2 + y^2 - ty$. Thus, the all solutions must satisfy $t^2+y^2-ty = C$ for some constant C. Using the initial condition, $y(1) = 3$, we have $C = 1+9-3 = 7$. So, the solution to the IVP satisfies $t^2 + y^2 - ty = 7$.

To draw the solution curve, rewrite the equation as $y^2 - ty + (t^2 - 7) = 0$. The left hand side is a quadratic equation in y, for which we seek roots. Using the quadratic formula, we have

$$\begin{aligned} y &= \frac{-(-t) \pm \sqrt{(-t)^2 - 4(1)(t^2-7)}}{2(1)} \\ &= \frac{t \pm \sqrt{-3t^2 + 28}}{2} \end{aligned}$$

Since our solution must satisfy $y(1) = 3$, we must take the positive square root. Then the solution is

$$y = \frac{t + \sqrt{-3t^2 + 28}}{2}$$

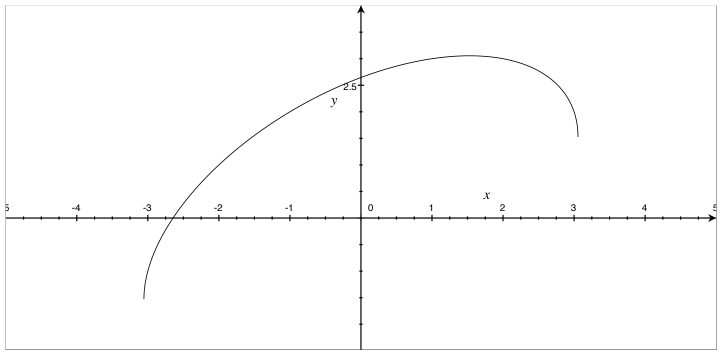

FIGURE 2

Problem 4. Solve the following ODEs by any method you have learned (if no initial condition is specified, find all solutions, possibly implicitly):

(i) $y\mathrm{d}t + t\mathrm{d}y = 0$ with initial condition $y(1) = 1$.

The equation is in the form $M\mathrm{d}t + N\mathrm{d}y = 0$ with $M(t, y) = y$ and $N(t, y) = t$. Since $M_y = 1 = N_t$, the equation is exact. So, there is a function $F = F(t, y)$ such that $\frac{\partial F}{\partial t} = M$ and $\frac{\partial F}{\partial y} = N$ and all solutions lie on the level curves of F.

Integrating M with respect to t,

$$\int \frac{\partial F}{\partial t} \mathrm{d}t = \int y \mathrm{d}t$$

$$F(t, y) = ty + f(y) \tag{7}$$

where $f(y)$ is an unknown function of y.

Integrating N with respect to y,

$$\int \frac{\partial F}{\partial y} \mathrm{d}y = \int t \mathrm{d}y$$

$$F(t, y) = ty + g(t) \tag{8}$$

where $g(t)$ is an unknown function of t.

Equating (7) and (8), we see that $F(t, y) = ty$. Thus, the all solutions must satisfy $ty = C$ for some constant C. Using the initial condition, $y(1) = 1$, we have $C = 1$. So, the solution to the IVP satisfies $ty = 1$.

(ii) $y' = \frac{2t+4y}{3y-4t}$.

Rewrite the equation as $(-2t-4y)\mathrm{d}t + (3y-4t)\mathrm{d}y = 0$. Then the equation is in the form $M\mathrm{d}t + N\mathrm{d}y = 0$ and $M_y = -4 = N_t$, so it is exact. Solving by the method outlined above, we obtain the general solution $\frac{3}{2}y^2 - t^2 - 4ty = C$.

(iii) $y' = \frac{2t+3}{2-2y}$ with initial condition $y(0) = 1$.

Rewrite the equation as $(-2t-3)\mathrm{d}t + (2-2y)\mathrm{d}y = 0$. Then the equation is in the form $M\mathrm{d}t + N\mathrm{d}y = 0$ and $M_y = 0 = N_t$, so it is exact. Solving by the method outlined above, we obtain the general solution $2y - y^2 - t^2 - 3t = C$. Using the initial condition, we find $C = 1$, so the solution to the IVP satisfies $2y - y^2 - t^2 - 3t = 1$.

(iv) $(9t^2 + y - 1)\mathrm{d}t + (t - 4y)\mathrm{d}y = 0$ with initial condition $y(1) = 0$.

The equation is in the form $M\mathrm{d}t + N\mathrm{d}y = 0$ and $M_y = 1 = N_t$, so it is exact. Solving by the method outlined above, we obtain the general solution $3t^3 - t - 2y^2 + ty = C$. Using the initial condition, we find $C = 2$, so the solution to the IVP satisfies $3t^3 - t - 2y^2 + ty = 2$.

Problem 5. Consider the ODE

$$(e^{2t} + y - 1)\mathrm{d}t - \mathrm{d}y = 0.$$

(i) Show that this ODE is not exact.

The equation is in the form $M\mathrm{d}t + N\mathrm{d}y = 0$ but $M_y = 1$ while $N_t = 0$. Since $M_y \neq N_t$, the ODE is not exact.

(ii) Find an integrating factor for the ODE.

Suppose $\mu = \mu(t)$ is an integrating factor for this ODE. Multiplying by μ,

$$\mu(t)(e^{2t} + y - 1)\mathrm{d}t - \mu(t)\mathrm{d}y = 0.$$

Let $\tilde{M} = \mu(t)(e^{2t} + y - 1)$ and $\tilde{N} = -\mu(t)$. For the equation to be exact, we need $\tilde{M}_y = \tilde{N}_t$. We compute $\tilde{M}_y = \mu(t)$ and $\tilde{N}_t = -\mu'(t)$. Setting these expressions equal, we obtain an ODE for μ: $\mu'(t) = -\mu(t)$. This equation is separable and a solution is $\mu(t) = e^{-t}$. Hence, we have the following exact ODE:

$$e^{-t}(e^{2t} + y - 1)\mathrm{d}t - e^{-t}\mathrm{d}y = 0.$$

or

$$(e^t + e^{-t}y - e^{-t})\mathrm{d}t - e^{-t}\mathrm{d}y = 0.$$

Solving this equation by the method outlined in problems 1 through 4, the general solution satisfies $-e^{-t}y + e^t + e^{-t} = C$ or $y = e^{2t} + 1 + Ce^t$.

Problem 6. Solve the ODE $y\mathrm{d}t + (2t - ye^y)\mathrm{d}y = 0$.

The equation is in the form $M\mathrm{d}t + N\mathrm{d}y = 0$ but $M_y = 1$ while $N_t = 2$. Since $M_y \neq N_t$, the ODE is not exact. We look for an integrating factor to transform the ODE into an exact equation. Suppose $\mu = \mu(y)$ is an integrating factor for this ODE. Multiplying by μ,

$$\mu(y)y\mathrm{d}t + \mu(y)(2t - ye^y)\mathrm{d}y = 0.$$

Let $\tilde{M} = \mu(y)y$ and $\tilde{N} = \mu(y)(2t - ye^y) = 2t\mu(y) - \mu(y)ye^y$. For the equation to be exact, we need $\tilde{M}_y = \tilde{N}_t$. We compute $\tilde{M}_y = \mu(y) + y\mu'(y)$ and $\tilde{N}_t = 2\mu(y)$. Setting these expressions equal, we obtain an ODE for μ: $\mu(y) + y\mu'(y) = 2\mu(y)$. Solving for $\mu'(y)$, we have $\mu'(y) = \frac{\mu}{y}$. This equation is separable and a solution is $\mu = y$. Hence, we have the following exact ODE:

$$y^2\mathrm{d}t + y(2t - ye^y)\mathrm{d}y = 0.$$

or

$$y^2\mathrm{d}t + (2ty - y^2e^y)\mathrm{d}y = 0.$$

Solving this equation by the method outlined in problems 1 through 4, the general solution satisfies $-ty^2 - (y^2 - 2y + 2)e^y = C$.