HOMEWORK 4, M 331
DUE 3/5/09

Problem 1. Consider the linear (inhomogeneous) ODE

$$y' + y = \sin t$$

(i) Find all solutions to the homogeneous ODE.

The associated homogeneous equation is

$$\frac{\mathrm{d}y_h}{\mathrm{d}t} + y_h = 0.$$

Separating variables,

$$\begin{aligned}
\frac{\mathrm{d}y_h}{\mathrm{d}t} &= -y_h \\
\frac{1}{y_h}\mathrm{d}y_h &= -1\mathrm{d}t \\
\int \frac{1}{y_h}\mathrm{d}y_h &= \int -1\mathrm{d}t \\
\ln|y_h| &= -t + C \\
e^{\ln|y_h|} &= e^{-t+C} \\
y_h &= e^{-t}e^{C} \\
y_h &= Ce^{-t},
\end{aligned}$$

where we have replaced e^C with C. So, all solutions to the homogeneous equation are $y_h = Ce^{-t}$, for arbitrary constants C.

(ii) Find one particular solution of the inhomogeneous ODE.

We look for a solution of the form $y_p = u(t)y_h$, where $u(t)$ is some function of t, and y_h is any solution of the homogeneous equation. We take $y_h = e^{-t}$. Since y_p satisfies the ODE, we have $y_p' + y_p = \sin t$.

Differentiating,

$$\begin{aligned}
y_p' &= \frac{\mathrm{d}}{\mathrm{d}t}\left[u(t)e^{-t}\right] \\
&= u'(t)e^{-t} - u(t)e^{-t} \qquad \text{by the product rule.}
\end{aligned}$$

Substituting into the ODE, we have

$$\begin{aligned}
u'(t)e^{-t} - u(t)e^{-t} + u(t)e^{-t} &= \sin t \\
u'(t)e^{-t} &= \sin t \\
u'(t) &= e^{t}\sin t
\end{aligned}$$

Integrating,

$$\int u'(t)\mathrm{d}t = \int e^{t}\sin t\mathrm{d}t$$

$$u(t) = \frac{e^t}{2}(\sin t - \cos t)$$

Since $y_p = u(t)y_h$,

$$y_p = \frac{e^t}{2}(\sin t - \cos t)e^{-t}$$

Thus, $y_p = \frac{1}{2}(\sin t - \cos t)$ is one solution of the ODE.

(iii) Write down all solutions of the ODE.

All solutions are of the form $y = y_p + Cy_h$, for some constant C. So, all solutions are $y = \frac{1}{2}(\sin t - \cos t) + Ce^{-t}$.

(iv) What happens to the solutions when $t \to \infty$?

As $t \to \infty$, $Ce^{-t} \to 0$ for any value of C. Thus, as $t \to \infty$, all solutions tend toward the particular solution $y = \frac{1}{2}(\sin t - \cos t)$.

(v) Find the solution which satisfies $y(0) = 0$.

Setting $t = 0$ and $y = 0$ in $y = \frac{1}{2}(\sin t - \cos t) + Ce^{-t}$, we have $0 = -\frac{1}{2} + C$. So, $C = \frac{1}{2}$ and $y = \frac{1}{2}(\sin t - \cos t) + \frac{1}{2}e^{-t}$.

Problem 2. Find the solution to $ty' + 2y = t^2 - t + 1$ with initial condition $y(1) = 1/2$.

First we rewrite the equation in the form $y' + a(t)y = b(t)$. Dividing by t, we obtain

$$y' + \frac{2}{t}y = t - 1 + \frac{1}{t}$$

We will solve the equation by the method of variation of parameters. First we solve the associated homogeneous equation $y_h' + \frac{2}{t}y_h = 0$.

Separating variables,

$$\begin{aligned}
\frac{\mathrm{d}y_h}{\mathrm{d}t} &= -\frac{2}{t}y_h \\
\frac{1}{y_h}\mathrm{d}y_h &= -\frac{2}{t}\mathrm{d}t \\
\int \frac{1}{y_h}\mathrm{d}y_h &= \int -\frac{2}{t}\mathrm{d}t \\
\ln|y_h| &= -2\ln|t| + C \\
\ln|y_h| &= \ln t^{-2} + C \\
e^{\ln|y_h|} &= e^{\ln t^{-2} + C} \\
y_h &= t^{-2}e^C \\
y_h &= Ct^{-2},
\end{aligned}$$

where we have replaced e^C with C. So, all solutions to the homogeneous equation are $y_h = Ct^{-2}$, for arbitrary constants C.

Next, we find a particular solution of the form $y_p = u(t)y_h$, using $y_h = t^{-2}$. Since

y_p satisfies the ODE, we have $y_p' + \frac{2}{t} y_p = t - 1 + \frac{1}{t}$.

Differentiating,

$$\begin{aligned} y_p' &= \frac{\mathrm{d}}{\mathrm{d}t}\left[u(t)t^{-2}\right] \\ &= u'(t)t^{-2} - 2u(t)t^{-3} \qquad \text{by the product rule.} \end{aligned}$$

Substituting into the ODE, we have

$$\begin{aligned} u'(t)t^{-2} - 2u(t)t^{-3} + \frac{2}{t}u(t)t^{-2} &= t - 1 + \frac{1}{t} \\ u'(t)t^{-2} &= t - 1 + \frac{1}{t} \\ u'(t) &= t^3 - t^2 + t \end{aligned}$$

Integrating,

$$\begin{aligned} \int u'(t)\mathrm{d}t &= \int (t^3 - t^2 + t)\mathrm{d}t \\ u(t) &= \frac{t^4}{4} - \frac{t^3}{3} + \frac{t^2}{2} \end{aligned}$$

Since $y_p = u(t)y_h$,

$$y_p = \left(\frac{t^4}{4} - \frac{t^3}{3} + \frac{t^2}{2}\right)t^{-2}$$

Thus, $y_p = \frac{t^2}{4} - \frac{t}{3} + \frac{1}{2}$ is one solution of the ODE.

All solutions are of the form $y = y_p + Cy_h$, for some constant C. So, all solutions are $y = \frac{t^2}{4} - \frac{t}{3} + \frac{1}{2} + Ct^{-2}$.

To solve the IVP, $y(1) = 1/2$, we substitute $t = 1$ and $y = 1/2$. We have

$$\frac{1}{2} = \frac{1}{4} - \frac{1}{3} + \frac{1}{2} + C.$$

So, $C = \frac{1}{12}$ and the solution is $y = \frac{t^2}{4} - \frac{t}{3} + \frac{1}{2} + \frac{1}{12t^2}$.

Problem 3. Find all solutions of the ODE $y' + 2ty = 2te^{-t^2}$.

This is a first order linear ODE, which we solve by the method of variation of parameters. First we solve the associated homogeneous equation $y_h' + 2ty_h = 0$.

Separating variables,

$$\begin{aligned} \frac{\mathrm{d}y_h}{\mathrm{d}t} &= -2ty_h \\ \frac{1}{y_h}\mathrm{d}y_h &= -2t\mathrm{d}t \\ \int \frac{1}{y_h}\mathrm{d}y_h &= \int -2t\mathrm{d}t \\ \ln|y_h| &= -t^2 + C \end{aligned}$$

$$\begin{aligned} e^{\ln|y_h|} &= e^{-t^2+C} \\ y_h &= e^{-t^2}e^C \\ y_h &= Ce^{-t^2}, \end{aligned}$$

where we have replaced e^C with C. So, all solutions to the homogeneous equation are $y_h = Ce^{-t^2}$, for arbitrary constants C.

Next, we find a particular solution of the form $y_p = u(t)y_h$, using $y_h = e^{-t^2}$. Since y_p satisfies the ODE, we have $y_p' + 2ty_p = 2te^{-t^2}$.

Differentiating,

$$\begin{aligned} y_p' &= \frac{\mathrm{d}}{\mathrm{d}t}\left[u(t)e^{-t^2}\right] \\ &= u'(t)e^{-t^2} - 2tu(t)e^{-t^2} \qquad \text{by the product rule.} \end{aligned}$$

Substituting into the ODE, we have

$$\begin{aligned} u'(t)e^{-t^2} - 2tu(t)e^{-t^2} + 2tu(t)e^{-t^2} &= 2te^{-t^2} \\ u'(t)e^{-t^2} &= 2te^{-t^2} \\ u'(t) &= 2t \end{aligned}$$

Integrating,

$$\begin{aligned} \int u'(t)\mathrm{d}t &= \int 2t\mathrm{d}t \\ u(t) &= t^2 \end{aligned}$$

Since $y_p = u(t)y_h$,

$$y_p = t^2e^{-t^2}$$

All solutions are of the form $y = y_p + Cy_h$, for some constant C. So, all solutions are $y = t^2e^{-t^2} + Ce^{-t^2}$.

Problem 4. Find a value for y_0 so that the solution of the ODE $y' - y = 1 + \sin t$ with $y(0) = y_0$ remains bounded as $t \to \infty$.

This is a first order linear ODE, which we solve by the method of variation of parameters. First we solve the associated homogeneous equation $y_h' - y_h = 0$.

Separating variables,

$$\begin{aligned} \frac{\mathrm{d}y_h}{\mathrm{d}t} &= y_h \\ \frac{1}{y_h}\mathrm{d}y_h &= \mathrm{d}t \\ \int \frac{1}{y_h}\mathrm{d}y_h &= \int \mathrm{d}t \\ \ln|y_h| &= t + C \end{aligned}$$

$$\begin{aligned} e^{\ln|y_h|} &= e^{t+C} \\ y_h &= e^t e^C \\ y_h &= Ce^t, \end{aligned}$$

where we have replaced e^C with C. So, all solutions to the homogeneous equation are $y_h = Ce^t$, for arbitrary constants C.

Next, we find a particular solution of the form $y_p = u(t)y_h$, using $y_h = e^t$. Since y_p satisfies the ODE, we have $y_p' - y_p = 1 + \sin t$.

Differentiating,

$$\begin{aligned} y_p' &= \frac{\mathrm{d}}{\mathrm{d}t}\left[u(t)e^t\right] \\ &= u'(t)e^t + u(t)e^t \qquad \text{by the product rule.} \end{aligned}$$

Substituting into the ODE, we have

$$\begin{aligned} u'(t)e^t + u(t)e^t - u(t)e^t &= 1 + \sin t \\ u'(t)e^t &= 1 + \sin t \\ u'(t) &= e^{-t} + e^{-t}\sin t \end{aligned}$$

Integrating,

$$\begin{aligned} \int u'(t)\mathrm{d}t &= \int (e^{-t} + e^{-t}\sin t)\mathrm{d}t \\ u(t) &= -e^{-t} + -\frac{e^{-t}}{2}(\sin t + \cos t) \end{aligned}$$

Since $y_p = u(t)y_h$,

$$y_p = \left[-e^{-t} + -\frac{e^{-t}}{2}(\sin t + \cos t)\right]e^t$$

Thus, $y_p = -1 + -\frac{1}{2}(\sin t + \cos t)$ is one solution of the ODE.

All solutions are of the form $y = y_p + Cy_h$, for some constant C. So, all solutions are $y = -1 + -\frac{1}{2}(\sin t + \cos t) + Ce^t$.

As $t \to \infty$, $Ce^t \to \infty$ for $C > 0$ and $Ce^t \to -\infty$ for $C < 0$. On the other hand,

$$\begin{aligned} \left|-1 + -\frac{1}{2}(\sin t + \cos t)\right| &\le |-1| + \left|-\frac{1}{2}\sin t\right| + \left|-\frac{1}{2}\cos t\right| \qquad \text{by the triangle inequality} \\ &= 1 + \frac{1}{2}|\sin t| + \frac{1}{2}|\cos t| \\ &\le 1 + \frac{1}{2} + \frac{1}{2} \qquad \text{since } |\sin t| \le 1 \text{ and } |\cos t| \le 1 \\ &= 2. \end{aligned}$$

So, the solution stays bounded if and only if $C = 0$. Substituting $C = 0$, $t = 0$ and $y = y_0$ into the solution $y = -1 + -\frac{1}{2}(\sin t + \cos t) + Ce^t$, we have $y_0 = -1 - \frac{1}{2} = -\frac{3}{2}$.

Problem 5. Solve the following ODEs (if an initial condition is given, find the solution satisfying this condition):

For problem 5, use the general method outlined in problems 1 through 4. The answers are given below.

(i) $y' - 2y = t^2 e^{2t}\,, \qquad y(0) = 0$

The solution to the homogeneous equation is $y_h = Ce^{2t}$.

A particular solution is $y_p = \frac{1}{3}t^3 e^{2t}$.

The general solution is $y = \frac{1}{3}t^3 e^{2t} + Ce^{2t}$.

The solution to the IVP is $y = \frac{1}{3}t^3 e^{2t}$.

(ii) $ty' + (t+1)y = t\,, \qquad y(\ln 2) = 1$

The solution to the homogeneous equation is $y_h = \frac{C}{t}e^{-t}$.

A particular solution is $y_p = 1 - \frac{1}{t}$.

The general solution is $y = 1 - \frac{1}{t} + \frac{C}{t}e^{-t}$.

The solution to the IVP is $y = 1 - \frac{1}{t} + \frac{2}{t}e^{-t}$.

(iii) $y' + y = \frac{1}{1+e^t}$

The solution to the homogeneous equation is $y_h = Ce^{-t}$.

A particular solution is $y_p = e^{-t}\ln(1+e^t)$.

The general solution is $y = e^{-t}\ln(1+e^t) + Ce^{-t}$.