HOMEWORK 5, M 331
DUE 3/12/09

Problem 1. Consider the linear 2nd order homogeneous ODE

$$y'' + 5y' + 6y = 0$$

(i) Assuming there is a solution of the form $y(t) = e^{\lambda t}$ find the condition on λ such that this $y(t)$ solves the ODE.

$$y' = \lambda e^{\lambda t}$$
$$y'' = \lambda^2 e^{\lambda t}$$

Substituting into the ODE,

$$\lambda^2 e^{\lambda t} + 5\lambda e^{\lambda t} + 6e^{\lambda t} = 0$$

Dividing by $e^{\lambda t}$,

$$\lambda^2 + 5\lambda + 6 = 0$$
$$(\lambda + 2)(\lambda + 3) = 0$$
$$\lambda = -2 \qquad \text{or} \qquad \lambda = -3$$

(ii) Write down the general solution of the ODE.

From part i, the fundamental solutions to the ODE are $y_1 = e^{-2t}$ and $y_2 = e^{-3t}$. So, the general solution is

$$y = c_1 e^{-2t} + c_2 e^{-3t}$$

for arbitrary constants c_1 and c_2.

(iii) Find the solution which satisfies the initial conditions $y(0) = y'(0) = 1$.

$$y = c_1 e^{-2t} + c_2 e^{-3t} \quad (1)$$
$$y' = -2c_1 e^{-2t} - 3c_2 e^{-3t} \quad (2)$$

Substituting $t = 0$ and $y = 1$ in the general solution (1), we obtain the equation

$$c_1 + c_2 = 1$$

Substituting $t = 0$ and $y' = 1$ into the derivative (2), we get

$$-2c_1 - 3c_2 = 1$$

To find the solution to the IVP, we must solve the system of equations

$$c_1 + c_2 = 1$$
$$-2c_1 - 3c_2 = 1$$

Multiplying the first equation by 2,

$$2c_1 + 2c_2 = 2$$
$$-2c_1 - 3c_2 = 1$$

Adding the equations,

$$-c_2 = 3$$
$$c_2 = -3$$

Substituting back into the equation $c_1 + c_2 = 1$,

$$c_1 - 3 = 1$$
$$c_1 = 4$$

Therefore, the desired solution is $y = 4e^{-2t} - 3e^{-3t}$.

(iv) Can you find a formula for the solution which satisfies the initial condition $y(0) = y_0$ and $y'(0) = v_0$? Once you have such a formula plug in the previous case $y_0 = v_0 = 1$ and check if you got the same answer.

We follow the same steps as in part iii.

$$y = c_1e^{-2t} + c_2e^{-3t} \tag{3}$$
$$y' = -2c_1e^{-2t} - 3c_2e^{-3t} \tag{4}$$

Substituting $t = 0$ and $y = y_0$ in the general solution (3), we obtain the equation

$$c_1 + c_2 = y_0$$

Substituting $t = 0$ and $y' = v_0$ into the derivative (4), we get

$$-2c_1 - 3c_2 = v_0$$

To find the solution to the IVP, we must solve the system of equations

$$c_1 + c_2 = y_0$$
$$-2c_1 - 3c_2 = v_0$$

Multiplying the first equation by 2,

$$2c_1 + 2c_2 = 2y_0$$
$$-2c_1 - 3c_2 = v_0$$

Adding the equations,

$$-c_2 = 2y_0 + v_0$$
$$c_2 = -2y_0 - v_0$$

Substituting back into the equation $c_1 + c_2 = y_0$,

$$c_1 - 2y_0 - v_0 = y_0$$
$$c_1 = 3y_0 + v_0$$

Therefore, the desired solution is $y = (3y_0 + v_0)e^{-2t} - (2y_0 + v_0)e^{-3t}$.

To check, we set $y_0 = v_0 = 1$, and indeed obtain $c_1 = 4$ and $c_2 = -3$ as above.

Problem 2. Find all solutions of the ODE $2y'' - 3y' + y = 0$.

The characteristic equation is $2\lambda^2 - 3\lambda + 1 = 0$. Factoring,

$$2\lambda^2 - 3\lambda + 1 = 0$$
$$(2\lambda - 1)(\lambda - 1) = 0$$
$$\lambda = 1/2 \qquad \text{or} \qquad \lambda = 1$$

Then the fundamental solutions are $y_1 = e^{\frac{1}{2}t}$ and $y_2 = e^t$. So, all solutions are $y = c_1 e^{\frac{1}{2}t} + c_2 e^t$ for arbitrary constants c_1 and c_2.

Problem 3. Find the solution of the ODE $2y'' + y' - 4y = 0$ (the numbers come out quite ugly, so I replaced this equation in a later posting by $2y'' + 2y' - 4y = 0$, you can do either of them of course) with initial conditions $y(0) = 0$ and $y'(0) = 1$.

The characteristic equation is $2\lambda^2 + \lambda - 4 = 0$. We find the roots using the quadratic formula.

$$2\lambda^2 + \lambda - 4 = 0$$
$$\lambda = \frac{-1 \pm \sqrt{1 + 32}}{4}$$
$$= \frac{-1 \pm \sqrt{33}}{4}$$

Then the fundamental solutions are $y_1 = e^{\left(\frac{-1+\sqrt{33}}{4}\right)t}$ and $y_2 = e^{\left(\frac{-1-\sqrt{33}}{4}\right)t}$. So, all solutions are $y = c_1y_1 + c_2y_2$ for arbitrary constants c_1 and c_2. To satisfy the initial conditions, we must have $c_1 = \frac{2}{33}\sqrt{33}$ and $c_2 = -\frac{2}{33}\sqrt{33}$.

Problem 4. Solve the initial value problem

$$y'' - 4y = 0\,, \qquad y(0) = 1\,, \quad y'(0) = v_0$$

and determine v_0 in such a way that this solution approaches zero as $t \to \infty$.

The characteristic equation is $\lambda^2 - 4 = 0$. Factoring,

$$\lambda^2 - 4 = 0$$
$$(\lambda - 2)(\lambda + 2) = 0$$
$$\lambda = 2 \qquad \text{or} \qquad \lambda = -2$$

Then the fundamental solutions are $y_1 = e^{2t}$ and $y_2 = e^{-2t}$. So, all solutions are $y = c_1e^{2t} + c_2e^{-2t}$ for arbitrary constants c_1 and c_2. To satisfy the initial conditions, we must have $c_1 = \frac{v_0+2}{4}$ and $c_2 = \frac{2-v_0}{4}$.

Now we find a solution that approaches zero and $t \to \infty$. Since $c_2e^{-2t} \to 0$ as $t \to \infty$ for any constant c_2, it suffices to find v_0 so that $c_1 = 0$. So, we take $v_0 = -2$. Then the solution $y = e^{-2t}$ satisfies the desired conditions.

Problem 5. Consider the inhomogeneous ODE $y'' - 3y' + 2y = 1$.

(i) Guess one solution of this ODE (Hint: what are the constant solutions?)

Suppose $y_p = c$ is a constant solution. Then $y_p' = 0$ and $y_p'' = 0$. Since y_p is a solution, we must have $y_p'' - 3y_p' + 2y_p = 1$. Substituting, we get $2c = 1$ which implies $c = 1/2$. Thus, $y_p = 1/2$ is the only constant solution.

(ii) Find all solutions of the corresponding homogeneous ODE.

The corresponding homogeneous ODE is $y_h'' - 3y_h' + 2y_h = 0$

The characteristic equation is $\lambda^2 - 3\lambda + 2 = 0$. Factoring,

$$\lambda^2 - 3\lambda + 2 = 0$$
$$(\lambda - 1)(\lambda - 2) = 0$$
$$\lambda = 1 \qquad \text{or} \qquad \lambda = 2$$

Then the fundamental solutions are $y_1 = e^t$ and $y_2 = e^{2t}$. So, all solutions to the homogeneous equation are $y_h = c_1e^t + c_2e^{2t}$ for arbitrary constants c_1 and c_2.

(iii) Find all solutions of the inhomogeneous ODE. What happens to those solutions as $t \to \infty$?

All solutions to the inhomogeneous ODE are $y = y_p + y_h$ where y_h is the general solution to the homogeneous equation and y_p is any particular solution to the inhomogeneous equation. By part i, $y_p = 1/2$ is one solution. Thus, all solutions are $y = \frac{1}{2} + c_1e^t + c_2e^{2t}$. Since $e^t \to \infty$ and $e^{2t} \to \infty$ as $t \to \infty$, all solutions except the constant solution go to $\pm\infty$ as $t \to \infty$, where the sign depends on the signs of the constants c_1 and c_2.

(iv) Find the solution with initial condition $y(0) = y'(0) = 0$.

Putting $t = 0$ and $y = 0$ in $y = \frac{1}{2} + c_1 e^t + c_2 e^{2t}$, we have

$$\frac{1}{2} + c_1 + c_2 = 0$$

Putting $t = 0$ and $y' = 0$ into $y' = c_1 e^t + 2c_2 e^{2t}$, we have

$$c_1 + 2c_2 = 0$$

Solving this system, we get $c_1 = -1$ and $c_2 = \frac{1}{2}$. Thus, the desired solution is $y = \frac{1}{2} - e^t + \frac{1}{2}e^{2t}$.

Problem 6. Consider the ODE $y'' - 4y' + 5y = 0$.

(i) Assuming there is a solution of the form $y(t) = e^{\lambda t}$ determine all λ such that this $y(t)$ solves the ODE.

Assuming there is such a solution, we know λ must satisfy the characteristic equation

$$\lambda^2 - 4\lambda + 5 = 0$$

To find the roots of this equation, we use the quadratic formula.

$$\begin{aligned}\lambda &= \frac{4 \pm \sqrt{16-20}}{2} \\ &= \frac{4 \pm \sqrt{-4}}{2} \\ &= \frac{4 \pm \sqrt{4}\sqrt{-1}}{2} \\ &= \frac{4 \pm 2i}{2} \\ &= 2 \pm i\end{aligned}$$

(ii) Use Euler's Formula $e^{ix} = \cos x + i \sin x$ and the superposition principle (applied to the 2 solutions found in (i)) to show that $e^{2t} \sin t$ and $e^{2t} \cos t$ are solutions of this ODE.

By part i, $w_1 = e^{(2+i)t}$ and $w_2 = e^{(2-i)t}$ solve the ODE. We use Euler's formula to rewrite the solutions.

$$\begin{aligned}w_1 &= e^{(2+i)t} \\ &= e^{2t}e^{it} \\ &= e^{2t}(\cos t + i \sin t)\end{aligned}$$

$$\begin{aligned} w_2 &= e^{(2-i)t} \\ &= e^{2t}e^{-it} \\ &= e^{2t}[\cos(-t) + i\sin(-t)] \\ &= e^{2t}(\cos t - i \sin t) \end{aligned}$$

By the superposition principle any linear combination of w_1 and w_2 is also a solution of the ODE. Then $y_1 = \frac{1}{2}w_1 + \frac{1}{2}w_2 = e^{2t}\cos t$ and $y_2 = -\frac{1}{2}iw_1 + \frac{1}{2}iw_2 = e^{2t}\sin t$ are also solutions.

(iii) Determine the solution which satisfies the initial condition $y(0) = 1$ and $y'(0) = 0$.

From part ii, the general solution of the ODE is

$$y = c_1 e^{2t}\cos t + c_2 e^{2t}\sin t \tag{5}$$

Taking derivatives,

$$y' = e^{2t}[(2c_1 + c_2)\cos t + (2c_2 - c_1)\sin t] \tag{6}$$

Substituting $t = 0$ and $y = 1$ into (5), we find $c_1 = 1$. And putting $t = 0$ and $y' = 0$ into (6), we get $2c_1 + c_2 = 0$. Using that $c_1 = 1$, we have $c_2 = -2$. Therefore, the solution to the IVP is $y = e^{2t}\cos t + -2e^{2t}\sin t$.

Problem 7. Consider the inhomogeneous ODE $y'' + 6y' + 13y = 5$.

(i) Find a solution of this ODE (compare with Problem 5).

Suppose $y_p = c$ is a constant solution. Then $y_p' = 0$ and $y_p'' = 0$. Since y_p is a solution, we must have $y_p'' + 6y_p' + 13y_p = 5$. Substituting, we get $13c = 5$ which implies $c = 5/13$. Thus, $y_p = 5/13$ is a solution.

(ii) Find all solutions to the corresponding homogenous ODE.

The corresponding homogeneous ODE is $y_h'' + 6y_h' + 13y_h = 0$

The characteristic equation is $\lambda^2 + 6\lambda + 13 = 0$. We use the quadratic formula to solve for λ.

$$\begin{aligned}\lambda &= \frac{-6 \pm \sqrt{36-52}}{2} \\ &= \frac{-6 \pm \sqrt{-16}}{2} \\ &= \frac{-6 \pm \sqrt{16}\sqrt{-1}}{2} \\ &= \frac{-6 \pm 4i}{2} \\ &= -3 \pm 2i\end{aligned}$$

Then the fundamental solutions are $y_1 = e^{-3t}\cos(2t)$ and $y_2 = e^{-3t}\sin(2t)$. So, all solutions to the homogeneous equation are $y_h = c_1 e^{-3t}\cos(2t) + c_2 e^{-3t}\sin(2t)$ for arbitrary constants c_1 and c_2.

(iii) Find all solutions to the inhomogeneous ODE. How do the solutions behave for $t \to \infty$?

All solutions to the inhomogeneous ODE are $y = y_p + y_h$ where y_h is the general solution to the homogeneous equation and y_p is any particular solution to the inhomogeneous equation. By part i, $y_p = 5/13$ is one solution. Thus, all solutions are $y = \frac{5}{13} + c_1 e^{-3t}\cos(2t) + c_2 e^{-3t}\sin(2t)$. Since $e^{-3t}\cos(2t) \to 0$ and $e^{-3t}\sin t \to 0$ as $t \to \infty$ for any constants c_1 and c_2, all solutions approach the constant solution, $y_p = 5/13$, as $t \to \infty$.

(iv) Find the solution with initial conditions $y(0) = 0$ and $y'(0) = 1$.

Putting $t = 0$ and $y = 0$ in $y = \frac{5}{13} + c_1 e^{-3t}\cos(2t) + c_2 e^{-3t}\sin(2t)$, we have

$$\begin{aligned}\frac{5}{13} + c_1 &= 0 \\ c_1 &= -5/13\end{aligned}$$

Putting $t = 0$ and $y' = 1$ into $y' = e^{-3t}[(2c_2 - 3c_1)\cos(2t) - (2c_1 + 3c_2)\sin(2t)]$, we have

$$\begin{aligned}2c_2 - 3c_1 &= 1 \\ c_2 &= -1/13\end{aligned}$$

Thus, the desired solution is $y = \frac{5}{13} + -\frac{5}{13}e^{-3t}\cos(2t) - \frac{1}{13}e^{-3t}\sin(2t)$.