

HOMEWORK 6 (PRACTICE MIDTERM), M 331
DUE 3/26/09

Problem 1. Calculate the impact velocity of an object of mass 140 pounds dropped from a height of 30 feet, assuming the air drag coefficient is 1.

The differential equation for velocity is $\frac{dv}{dt} = 32.2 - \frac{\gamma}{m}v$, where $g = 32.2\text{ft/s}^2$ is the acceleration due to gravity, m is the mass of the object, and γ is the drag coefficient. Separating variables,

$$\begin{aligned}\int \frac{1}{32.2 - \frac{\gamma}{m}v} dv &= \int 1 dt \\ -\frac{m}{\gamma} \ln(32.2 - \frac{\gamma}{m}v) &= t + C \\ \ln(32.2 - \frac{\gamma}{m}v) &= -\frac{\gamma}{m}t + C \\ e^{\ln(32.2 - \frac{\gamma}{m}v)} &= e^{-\frac{\gamma}{m}t + C} \\ 32.2 - \frac{\gamma}{m}v &= e^{-\frac{\gamma}{m}t} e^C \\ -\frac{\gamma}{m}v &= C e^{-\frac{\gamma}{m}t} - 32.2 \\ v(t) &= C e^{-\frac{\gamma}{m}t} + \frac{32.2m}{\gamma}\end{aligned}$$

Setting $t = 0$, we have

$$v(0) = C + \frac{32.2m}{\gamma}$$

So, $C = v_0 - \frac{32.2m}{\gamma}$. The equation for velocity is then

$$v(t) = \left(v_0 - \frac{32.2m}{\gamma} \right) e^{-\frac{\gamma}{m}t} + \frac{32.2m}{\gamma}$$

Now we use that $v_0 = 0$, $m = 140/32.2$ (there is some confusion about this perhaps: pounds – as opposed to kilograms – refer to a force – weight – and thus to get the mass one has to divide by the gravitational constant 32.2, but even if you didn't or got it all confused, you would get full credit as long as the math is right and consistent) and $\gamma = 1$. The equation becomes

$$v(t) = -140e^{-0.23t} + 140$$

Now we need to find the equation for displacement. The differential equation is $\frac{ds}{dt} = v(t) = -140e^{-0.23t} + 140$. Integrating,

$$\int \frac{ds}{dt} dt = \int (-140e^{-0.23t} + 140) dt$$

$$s(t) = \frac{14000}{23} e^{-0.23t} + 140t + C$$

To find C , we use that $s(0) = 0$. Substituting $t = 0$ and $s = 0$, we get $C = -\frac{14000}{23}$. So, our equation is

$$s(t) = \frac{14000}{23} e^{-0.23t} + 140t - \frac{14000}{23}$$

We need to find the time of impact, i.e. when the displacement is 30 feet.

$$30 = \frac{14000}{23} e^{-0.23t} + 140t - \frac{14000}{23}$$

We solve for t graphically, and find $t \approx 1.44$. Putting this value into our equation for velocity, the impact velocity is approximately 39.5 ft/s.

Problem 2. Find the solution to the ODE $y' + t^2 y = 5t^2$ with initial condition $y(0) = 0$.

We can rewrite the equation as

$$\frac{dy}{dt} = t^2(5 - y)$$

and hence see it is separable. Separating variables,

$$\frac{1}{5 - y} dy = t^2 dt$$

Integrating,

$$\int \frac{1}{5 - y} dy = \int t^2 dt$$

$$-\ln(5 - y) = t^3/3 + C$$

$$\ln\left(\frac{1}{5 - y}\right) = t^3/3 + C$$

$$e^{\ln\left(\frac{1}{5 - y}\right)} = e^{t^3/3 + C}$$

$$\frac{1}{5 - y} = Ce^{t^3/3}$$

$$1 = C(5 - y)e^{t^3/3}$$

$$1 - 5Ce^{t^3/3} = -Cye^{t^3/3}$$

$$y = \frac{1 - 5Ce^{t^3/3}}{-Ce^{t^3/3}}$$

$$y = Ce^{-t^3/3} + 5$$

Setting $t = 0$ and $y = 0$, we find $C = -5$. Thus, the solution is $y = -5e^{-t^3/3} + 5$.

Problem 3. Solve the ODE $dx + (x/y - \sin y)dy = 0$.

The equation is in the form $Mdx + Ndy = 0$ but $M_y = 0$ while $N_x = 1/y$. Since $M_y \neq N_x$, the ODE is not exact. However, we note that the quotient $\frac{N_x - M_y}{M} = \frac{1}{y}$ is a function of y only. This suggests that we look for an integrating factor $\mu = \mu(y)$. Multiplying by μ ,

$$\mu(y)dx + \mu(y)(x/y - \sin y)dy = 0.$$

Let $\tilde{M} = \mu(y)$ and $\tilde{N} = \mu(y)(x/y - \sin y) = \mu(y)x/y - \mu(y)\sin y$. For the equation to be exact, we need $\tilde{M}_y = \tilde{N}_x$. We compute $\tilde{M}_y = \mu'(y)$ and $\tilde{N}_x = \mu(y)/y$. Setting these expressions equal, we obtain an ODE for μ : $\mu'(y) = \frac{\mu}{y}$. This equation is separable and a solution is $\mu = y$. Hence, we have the following exact ODE:

$$ydx + y(x/y - \sin y)dy = 0.$$

or

$$ydx + (x - y \sin y)dy = 0.$$

Now we check that $\tilde{M}_y = 1 = \tilde{N}_x$ and the equation is exact. So, there is a function $F = F(x, y)$ such that $\frac{\partial F}{\partial x} = \tilde{M}$ and $\frac{\partial F}{\partial y} = \tilde{N}$ and all solutions lie on the level curves of F .

Integrating \tilde{M} with respect to x ,

$$\begin{aligned} \int \frac{\partial F}{\partial x} dx &= \int y dx \\ (1) \quad F(x, y) &= xy + f(y) \end{aligned}$$

where $f(y)$ is an unknown function of y .

Integrating \tilde{N} with respect to y ,

$$\begin{aligned} \int \frac{\partial F}{\partial y} dy &= \int (x - y \sin y) dy \\ (2) \quad F(x, y) &= xy - \sin y + y \cos y + g(x) \end{aligned}$$

where $g(x)$ is an unknown function of x .

Equating (1) and (2), we see that $F(x, y) = xy - \sin y + y \cos y$. Thus, the all solutions must satisfy $xy - \sin y + y \cos y = C$ for some constant C .

Problem 4. Find the solution of the ODE $y'' - 2y' - 3y = 0$ with initial conditions $y(0) = 1$ and $y'(0) = 0$.

The characteristic equation is $\lambda^2 - 2\lambda - 3 = 0$. Factoring,

$$\begin{aligned}\lambda^2 - 2\lambda - 3 &= 0 \\ (\lambda - 3)(\lambda + 1) &= 0 \\ \lambda = 3 \quad \text{or} \quad \lambda &= -1\end{aligned}$$

Then the fundamental solutions are $y_1 = e^{3t}$ and $y_2 = e^{-t}$. So, all solutions are $y = c_1e^{3t} + c_2e^{-t}$ for arbitrary constants c_1 and c_2 .

Putting $t = 0$ and $y = 1$ in $y = c_1e^{3t} + c_2e^{-t}$, we have

$$c_1 + c_2 = 1$$

Putting $t = 0$ and $y' = 0$ into $y' = 3c_1e^{3t} - c_2e^{-t}$, we have

$$3c_1 - c_2 = 0$$

Solving this system, we get $c_1 = 1/4$ and $c_2 = 3/4$. Thus, the desired solution is $y = \frac{1}{4}e^{3t} + \frac{3}{4}e^{-t}$.

Problem 5. Find the general solution of the ODE $y'' + 2y' + 5y = 0$.

The characteristic equation is $\lambda^2 + 2\lambda + 5 = 0$. We use the quadratic formula to solve for λ .

$$\begin{aligned}\lambda &= \frac{-2 \pm \sqrt{4 - 20}}{2} \\ &= \frac{-2 \pm \sqrt{-16}}{2} \\ &= \frac{-2 \pm \sqrt{16}\sqrt{-1}}{2} \\ &= \frac{-2 \pm 4i}{2} \\ &= -1 \pm 2i\end{aligned}$$

Then the fundamental solutions are $y_1 = e^{-t} \cos(2t)$ and $y_2 = e^{-t} \sin(2t)$. So, the general solution is $y = c_1e^{-t} \cos(2t) + c_2e^{-t} \sin(2t)$ for arbitrary constants c_1 and c_2 .

Problem 6. Solve the ODE $4y'' + 4y' + y = 0$ with initial condition $y(0) = 0$ and $y'(0) = 1$.

The characteristic equation is $4\lambda^2 + 4\lambda + 1 = 0$. Factoring,

$$\begin{aligned}4\lambda^2 + 4\lambda + 1 &= 0 \\ (2\lambda + 1)(2\lambda + 1) &= 0 \\ \lambda &= -1/2\end{aligned}$$

Then the fundamental solutions are $y_1 = e^{-t/2}$ and $y_2 = te^{-t/2}$. So, all solutions are $y = c_1e^{-t/2} + c_2te^{-t/2}$ for arbitrary constants c_1 and c_2 .

Putting $t = 0$ and $y = 0$ in $y = c_1e^{-t/2} + c_2te^{-t/2}$, we find $c_1 = 0$.

Putting $t = 0$ and $y' = 1$ into $y' = -\frac{1}{2}c_1e^{-t/2} + c_2e^{-t/2} - \frac{1}{2}c_2te^{-t/2}$, we have

$$\begin{aligned} -\frac{1}{2}c_1 + c_2 &= 1 \\ c_2 &= 1 \end{aligned}$$

Thus, the desired solution is $y = te^{-t/2}$.