Homework 7, M 331
due 4/2/09

Problem 1. Consider the inhomogeneous ODE

$$y'' - 6y' + 10y = f(t).$$

(i) Solve the ODE with inhomogeneity $f(t) = e^{3t}$ subject to the initial condition $y(0) = 1$, $y'(0) = 3$.

The general solution is $y = y_p + y_h$ where y_h is the general solution to the homogeneous equation and y_p is any particular solution of the inhomogeneous equation.

First we find y_h by solving the homogeneous ODE $y_h'' - 6y_h' + 10y_h = 0$

The characteristic equation is $\lambda^2 - 6\lambda + 10 = 0$. We use the quadratic formula to solve for λ.

$$\begin{aligned} \lambda &= \frac{6 \pm \sqrt{36 - 40}}{2} \\ &= 3 \pm i \end{aligned}$$

Then the fundamental solutions are $y_1 = e^{3t}\cos t$ and $y_2 = e^{3t}\sin t$. So, $y_h = k_1 e^{3t}\cos t + k_2 e^{3t}\sin t$ for arbitrary constants k_1 and k_2.

We use variation of parameters to find a particular solution. We look for a solution of the form $y_p = c_1(t)y_1(t) + c_2(t)y_2(t)$ where $c_1(t)$ and $c_2(t)$ are unknown functions of t and y_1 and y_2 are the fundamental solutions of the homogeneous equation. We have the following formulas for c_1 and c_2:

$$\begin{aligned} c_1(t) &= -\int \frac{y_2 f}{W} \mathrm{d}t \\ c_2(t) &= \int \frac{y_1 f}{W} \mathrm{d}t \end{aligned}$$

where $W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$ is the Wronskian determinant.

We compute

$$\begin{aligned} y_1 &= e^{3t}\cos t \\ y_1' &= 3e^{3t}\cos t - e^{3t}\sin t \end{aligned}$$

$$\begin{aligned} y_2 &= e^{3t}\sin t \\ y_2' &= 3e^{3t}\sin t + e^{3t}\cos t \end{aligned}$$

$$
\begin{aligned}
W(t) &= y_1 y_2' - y_1' y_2 \\
&= e^{3t}\cos t(3e^{3t}\sin t + e^{3t}\cos t) - e^{3t}\sin t(3e^{3t}\cos t - e^{3t}\sin t) \\
&= e^{6t}(\sin^2 t + \cos^2 t) \\
&= e^{6t}
\end{aligned}
$$

Now we use our formulas to find c_1 and c_2.

$$
\begin{aligned}
c_1(t) &= -\int \frac{y_2 f}{W}\mathrm{d}t \\
&= -\int \frac{(e^{3t}\sin t)e^{3t}}{e^{6t}}\mathrm{d}t \\
&= -\int \sin t\mathrm{d}t \\
&= \cos t
\end{aligned}
$$

$$
\begin{aligned}
c_2(t) &= \int \frac{y_1 f}{W}\mathrm{d}t \\
&= \int \frac{(e^{3t}\cos t)e^{3t}}{e^{6t}}\mathrm{d}t \\
&= \int \cos t\mathrm{d}t \\
&= \sin t
\end{aligned}
$$

Then $y_p = \cos t(e^{3t}\cos t) + \sin t(e^{3t}\sin t = e^{3t}$. So, the general solution is $y = e^{3t} + k_1 e^{3t}\cos t + k_2 e^{3t}\sin t$. The solution to the initial value problem is $y = e^{3t}$.

(ii) Find the general solution of the ODE with inhomogeneity $f(t) = \sin t$.

The general solution is $y = y_p + y_h$ where y_h is the general solution to the homogeneous equation and y_p is any particular solution of the inhomogeneous equation.

First we find y_h by solving the homogeneous ODE $y_h'' - 6y_h' + 10y_h = 0$ From part i, the fundamental solutions are $y_1 = e^{3t}\cos t$ and $y_2 = e^{3t}\sin t$. So, $y_h = k_1 e^{3t}\cos t + k_2 e^{3t}\sin t$ for arbitrary constants k_1 and k_2.

We use the method of undetermined coefficients to find a particular solution y_p. Because the inhomogeneity is $f(t) = \sin t$, we guess that $y_p = A\cos t + B\sin t$ for some constants A and B. We use the ODE to find A and B. We compute,

$$
\begin{aligned}
y_p &= A\cos t + B\sin t \\
y_p' &= -A\sin t + B\cos t \\
y_p'' &= -A\cos t - B\sin t
\end{aligned}
$$

Since we want $y_p'' - 6y_p' + 10y_p = \sin t$, we have

$$(-A\cos t - B\sin t) - 6(-A\sin t + B\cos t) + 10(A\cos t + B\sin t) = \sin t$$

$$(9A - 6B)\cos t + (6A + 9B)\sin t = \sin t$$

Equating the coefficients of the sine and cosine terms on the left and right hand sides, we obtain the following system of equations

$$9A - 6B = 0$$

$$6A + 9B = 1$$

Solving this system yields $A = 2/39$ and $B = 1/13$. So, $y_p = \frac{2}{39}\cos t + \frac{1}{13}\sin t$. Therefore, the general solution is $y = \frac{2}{39}\cos t + \frac{1}{13}\sin t + k_1 e^{3t}\cos t + k_2 e^{3t}\sin t$.

(iii) Find a particular solution of the ODE with inhomogeneity $f(t) = e^{3t}\cos t$.

We use the method of undetermined coefficients to find a particular solution y_p. Because the inhomogeneity is $f(t) = e^{3t}\cos t$, we would guess that $y_p = Ae^{3t}\cos t + Be^{3t}\sin t$ for some constants A and B. However, this is a solution of the homogeneous equation. So, instead we guess $y_p = t(Ae^{3t}\cos t + Be^{3t}\sin t)$. We use the ODE to find A and B. We compute,

$$y_p = te^{3t}(A\cos t + B\sin t)$$

$$y_p' = e^{3t}(A\cos t + B\sin t) + te^{3t}[(3A + B)\cos t + (3B - A)\sin t]$$

$$y_p'' = e^{3t}[(6A + 2B)\cos t + (6B - 2A)\sin t] + te^{3t}[(8A + 6B)\cos t + (8B - 6A)\sin t]$$

Since we want $y_p'' - 6y_p' + 10y_p = e^{3t}\cos t$, we have

$$e^{3t}[2B\cos t - 2A\sin t] = e^{3t}\cos t$$

Equating coefficients, we obtain

$$2B = 1 \Rightarrow B = 1/2$$

$$-2A = 0 \Rightarrow A = 0$$

Then a particular solution is $y_p = \frac{1}{2}te^{3t}\sin t$.

Problem 2. Find the general solution of the inhomogeneous ODE

$$y'' + y' = 1 + e^{-t} + e^{t}.$$

The general solution is $y = y_p + y_h$ where y_h is the general solution to the homogeneous equation and y_p is any particular solution of the inhomogeneous equation.

First we find y_h by solving the homogeneous ODE $y_h'' + y_h' = 0$

The characteristic equation is $\lambda^2 + \lambda = 0$. Factoring,

$$\lambda^2 + \lambda = 0$$
$$\lambda(\lambda + 1) = 0$$
$$\lambda = 0 \qquad \text{or} \qquad \lambda = -1$$

Then the fundamental solutions are $y_1 = 1$ and $y_2 = e^{-t}$. So, $y_h = k_1 + k_2 e^{-t}$ for arbitrary constants k_1 and k_2.

We use variation of parameters to find a particular solution. We look for a solution of the form $y_p = c_1(t)y_1(t) + c_2(t)y_2(t)$ where $c_1(t)$ and $c_2(t)$ are unknown functions of t and y_1 and y_2 are the fundamental solutions of the homogeneous equation. We have the following formulas for c_1 and c_2:

$$c_1(t) = -\int \frac{y_2 f}{W} \mathrm{d}t$$
$$c_2(t) = \int \frac{y_1 f}{W} \mathrm{d}t$$

where $W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$ is the Wronskian determinant.

We compute

$$y_1 = 1$$
$$y_1' = 0$$

$$y_2 = e^{-t}$$
$$y_2' = -e^{-t}$$

$$W(t) = y_1 y_2' - y_1' y_2 = -e^{-t}$$

Now we use our formulas to find c_1 and c_2.

$$\begin{aligned}
c_1(t) &= -\int \frac{y_2 f}{W}\mathrm{d}t \\
&= -\int \frac{e^{-t}(1+e^{-t}+e^t)}{-e^{-t}}\mathrm{d}t \\
&= \int (1+e^{-t}+e^t)\mathrm{d}t \\
&= t - e^{-t} + e^t
\end{aligned}$$

$$\begin{aligned}
c_2(t) &= \int \frac{y_1 f}{W}\mathrm{d}t \\
&= \int \frac{(1+e^{-t}+e^t)}{-e^{-t}}\mathrm{d}t \\
&= \int -e^t - 1 - e^{2t}\mathrm{d}t \\
&= -e^t - t - \frac{1}{2}e^{2t}
\end{aligned}$$

Then $y_p = (t - e^{-t} + e^t) + (-e^t - t - \frac{1}{2}e^{2t})e^{-t} = t - e^{-t} + \frac{1}{2}e^t - te^{-t} - 1$. So, the general solution is $y = t - e^{-t} + \frac{1}{2}e^t - te^{-t} - 1 + k_1 + k_2 e^{-t}$.

Problem 3. Find a particular solution of the inhomogeneous ODE

$$y'' - 2y' + y = e^t + 3te^t + t^2$$

First we find y_h by solving the homogeneous ODE $y_h'' - 2y_h' + y_h = 0$

The characteristic equation is $\lambda^2 - 2\lambda + 1 = 0$. Factoring,

$$\begin{aligned}
\lambda^2 - 2\lambda + 1 &= 0 \\
(\lambda - 1)(\lambda - 1) &= 0 \\
\lambda &= 1
\end{aligned}$$

Then the fundamental solutions are $y_1 = e^t$ and $y_2 = te^t$.

We use variation of parameters to find a particular solution. We look for a solution of the form $y_p = c_1(t)y_1(t) + c_2(t)y_2(t)$ where $c_1(t)$ and $c_2(t)$ are unknown functions of t and y_1 and y_2 are the fundamental solutions of the homogeneous equation. We have the following formulas for c_1 and c_2:

$$\begin{aligned}
c_1(t) &= -\int \frac{y_2 f}{W}\mathrm{d}t \\
c_2(t) &= \int \frac{y_1 f}{W}\mathrm{d}t
\end{aligned}$$

where $W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$ is the Wronskian determinant.

We compute

$$y_1 = e^t$$
$$y_1' = e^t$$

$$y_2 = te^t$$
$$y_2' = (1+t)e^t$$

$$W(t) = y_1 y_2' - y_1' y_2 = (1+t)e^{2t} - te^{2t} = e^{2t}$$

Now we use our formulas to find c_1 and c_2.

$$\begin{aligned} c_1(t) &= -\int \frac{y_2 f}{W} \mathrm{d}t \\ &= -\int \frac{te^t(e^t + 3te^t + t^2)}{e^{2t}} \mathrm{d}t \\ &= \int (-t - 3t^2 - t^3 e^{-t}) \mathrm{d}t \\ &= -\frac{1}{2}t^2 - t^3 + e^{-t}(t^3 + 3t^2 + 6t + 6) \end{aligned}$$

$$\begin{aligned} c_2(t) &= \int \frac{y_1 f}{W} \mathrm{d}t \\ &= \int \frac{e^t(e^t + 3te^t + t^2)}{e^{2t}} \mathrm{d}t \\ &= \int 1 + 3t + t^2 e^{-t} \mathrm{d}t \\ &= t + \frac{3}{2}t^2 - e^{-t}(t^2 + 2t + 2) \end{aligned}$$

Then $y_p = t^2 + 4t + 6 + \frac{1}{2}t^2 e^t + \frac{1}{2}t^3 e^t$

Problem 4. Find the solution of the inhomogeneous ODE

$$y'' + 6y' + 9y = t^2$$

with initial conditions $y(0) = y'(0) = 0$.

The general solution is $y = y_p + y_h$ where y_h is the general solution to the homogeneous equation and y_p is any particular solution of the inhomogeneous equation.

First we find y_h by solving the homogeneous ODE $y_h'' + 6y_h' + 9y_h = 0$

The characteristic equation is $\lambda^2 + 6\lambda + 9 = 0$. Factoring,

$$\lambda^2 + 6\lambda + 9 = 0$$
$$(\lambda + 3)(\lambda + 3) = 0$$
$$\lambda = -3$$

Then the fundamental solutions are $y_1 = e^{-3t}$ and $y_2 = te^{-3t}$. So, $y_h = k_1 e^{-3t} + k_2 te^{-3t}$ for arbitrary constants k_1 and k_2.

We use the method of undetermined coefficients to find a particular solution y_p. Because the inhomogeneity is $f(t) = t^2$, we guess that $y_p = At^2 + Bt + C$ for some constants A, B, and C. We use the ODE to find A, B and C. We compute,

$$y_p = At^2 + Bt + C$$
$$y_p' = 2At + B$$
$$y_p'' = 2A$$

Since we want $y_p'' + 6y_p' + 9y_p = t^2$, we have

$$2A + 6(2At + B) + 9(At^2 + Bt + C) = t^2$$
$$9At^2 + (12A + 9B)t + (2A + 6B + 9C) = t^2$$

Equating coefficients, we obtain the following system of equations

$$9A = 1$$
$$12A + 9B = 0$$
$$2A + 6B + 9C = 0$$

Solving this system yields $A = 1/9$, $B = -4/27$ and $C = 2/27$. So, $y_p = \frac{1}{9}t^2 - \frac{4}{27}t + \frac{2}{27}$. Therefore, the general solution is $y = \frac{1}{9}t^2 - \frac{4}{27}t + \frac{2}{27} + k_1 e^{-3t} + k_2 te^{-3t}$. To satisfy the IVP we need $k_1 = -2/27$ and $k_2 = 4/27$.