HOMEWORK 8, M 331
DUE 4/9/09

Problem 1. Consider the inhomogeneous ODE

$$t^2y'' - 2y = 3t^2 - 1\,.$$

(i) Show that $y_1(t) = t^2$ and $y_2(t) = 1/t$ are solutions to the homogeneous ODE.

We need to show that $t^2y_i'' - 2y_i = 0$ for $i = 1, 2$.

$$\begin{aligned} y_1(t) &= t^2 \\ y_1'(t) &= 2t \\ y_1''(t) &= 2 \end{aligned}$$

Then

$$t^2y_1'' - 2y_1 = t^2(2) - 2t^2 = 0$$

$$\begin{aligned} y_2(t) &= 1/t \\ y_2'(t) &= -1/t^2 \\ y_2''(t) &= 2/t^3 \end{aligned}$$

$$t^2y_2'' - 2y_2 = t^2(2/t^3) - 2(1/t) = 2/t - 2/t = 0$$

(ii) Verify that y_1 and y_2 are independent solutions.

We need to show that $W(t) = y_1y_2' - y_1'y_2 \neq 0$.

$$W(t) = t^2(-1/t^2) - 2t(1/t) = -1 - 2 = -3 \neq 0$$

So, y_1 and y_2 are independent.

(iii) Find a particular solution of the ODE using the variation of parameter method.

Since y_1 and y_2 are independent solutions of the homogeneous equation, we can use the method of variation of parameters to find a particular solution of the inhomogeneous equation. We look for solution of the form $y_p(t) = c_1(t)y_1(t) + c_2(t)y_2(t)$ where c_1 and c_2 are given by the following formulas:

$$\begin{aligned} c_1(t) &= -\int \frac{y_2 f}{W}\mathrm{d}t \\ c_2(t) &= \int \frac{y_1 f}{W}\mathrm{d}t \end{aligned}$$

where W is the Wronskian, y_1 and y_2 are a fundamental set of solutions to the homogeneous equation and f is the inhomogeneity when the equation is in the form $y'' + a(t)y' + b(t)y = f(t)$. Dividing by t^2, the equation becomes $y'' - (2/t^2)y = 3 - 1/t^2$.

From part ii, $W(t) = -3$, so

$$\begin{aligned} c_1(t) &= -\int \frac{y_2 f}{W} \mathrm{d}t \\ &= -\int \frac{(1/t)(3 - 1/t^2)}{-3} \mathrm{d}t \\ &= \frac{1}{3} \int \left(\frac{3}{t} - \frac{1}{t^3} \right) \mathrm{d}t \\ &= \ln(t) + \frac{1}{6t^2} \end{aligned}$$

$$\begin{aligned} c_2(t) &= \int \frac{y_1 f}{W} \mathrm{d}t \\ &= \int \frac{t^2(3 - 1/t^2)}{-3} \mathrm{d}t \\ &= \int \left(-t^2 + \frac{1}{3} \right) \mathrm{d}t \\ &= -\frac{1}{3}t^3 + \frac{1}{3}t \end{aligned}$$

Then $y_p = \left(\ln(t) + \frac{1}{6t^2}\right) t^2 + \left(-\frac{1}{3}t^3 + \frac{1}{3}t\right)(1/t) = t^2 \ln(t) - \frac{1}{3}t^2 + \frac{1}{2}$

Problem 2. Consider the inhomogeneous ODE

$$t^2 y'' - 3ty' + 4y = t^2 \ln t\,.$$

(i) Show that $y_1(t) = t^2$ and $y_2(t) = t^2 \ln t$ are solutions to the homogeneous ODE.

We need to show that $t^2 y_i'' - 2y_i = 0$ for $i = 1, 2$.

$$\begin{aligned} y_1(t) &= t^2 \\ y_1'(t) &= 2t \\ y_1''(t) &= 2 \end{aligned}$$

Then

$$t^2 y_1'' - 3ty_1' + 4y_1 = t^2(2) - 3t(2t) + 4(t^2) = 0$$

$$\begin{aligned} y_2(t) &= t^2 \ln t \\ y_2'(t) &= 2t \ln t + t \end{aligned}$$

$$y_2''(t) = 2\ln t + 3$$

$$t^2 y_2'' - 3ty_2' + 4y_2 = t^2(2\ln t + 3) - 3t(2t\ln t + t) + 4(t^2 \ln t) = 0$$

(ii) Verify that y_1 and y_2 are independent solutions.

We need to show that $W(t) = y_1 y_2' - y_1' y_2 \neq 0$.

$$W(t) = t^2(2t\ln t + t) - 2t(t^2 \ln t) = t^3$$

From the differential equation, t must be greater than zero. So, $W \neq 0$.

(iii) Find a particular solution of the ODE using the variation of parameter method.

Since y_1 and y_2 are independent solutions of the homogeneous equation, we can use the method of variation of parameters to find a particular solution of the inhomogeneous equation. We look for solution of the form $y_p(t) = c_1(t)y_1(t) + c_2(t)y_2(t)$ where c_1 and c_2 are given by the following formulas:

$$c_1(t) = -\int \frac{y_2 f}{W} \mathrm{d}t$$

$$c_2(t) = \int \frac{y_1 f}{W} \mathrm{d}t$$

where W is the Wronskian, y_1 and y_2 are a fundamental set of solutions to the homogeneous equation and f is the inhomogeneity when the equation is in the form $y'' + a(t)y' + b(t)y = f(t)$. Dividing by t^2, the equation becomes $y'' - (3/t)y' + (4/t^2)y = \ln t$.

From part ii, $W(t) = t^3$, so

$$\begin{aligned} c_1(t) &= -\int \frac{y_2 f}{W} \mathrm{d}t \\ &= -\int \frac{(t^2 \ln t)\ln t}{t^3} \mathrm{d}t \\ &= -\int \frac{(\ln t)^2}{t} \mathrm{d}t \\ &= -\frac{1}{3}(\ln t)^3 \end{aligned}$$

$$\begin{aligned}
c_2(t) &= \int \frac{y_1 f}{W} \mathrm{d}t \\
&= \int \frac{t^2 \ln t}{t^3} \mathrm{d}t \\
&= \int \frac{\ln t}{t} \mathrm{d}t \\
&= \frac{1}{2} (\ln t)^2
\end{aligned}$$

Then $y_p = -\frac{1}{3}(\ln t)^3 t^2 + \frac{1}{2}(\ln t)^2 t^2 \ln t = \frac{1}{6} t^2 (\ln t)^3$

Problem 3. Find the general solution of the inhomogeneous ODE

$$y'' + 4y' + 4y = \frac{1}{t^2} e^{-2t} .$$

The general solution is $y = y_p + y_h$ where y_h is the general solution to the homogeneous equation and y_p is any particular solution of the inhomogeneous equation.

First we find y_h by solving the homogeneous ODE $y_h'' + 4y_h' + 4y_h = 0$

The characteristic equation is $\lambda^2 + 4\lambda + 4 = 0$. Factoring,

$$\begin{aligned}
\lambda^2 + 4\lambda + 4 &= 0 \\
(\lambda + 2)(\lambda + 2) &= 0 \\
\lambda &= -2
\end{aligned}$$

Then the fundamental solutions are $y_1 = e^{-2t}$ and $y_2 = te^{-2t}$. So, $y_h = k_1 e^{-2t} + k_2 t e^{-2t}$ for arbitrary constants k_1 and k_2.

We use variation of parameters to find a particular solution. We look for a solution of the form $y_p = c_1(t) y_1(t) + c_2(t) y_2(t)$ where $c_1(t)$ and $c_2(t)$ are unknown functions of t and y_1 and y_2 are the fundamental solutions of the homogeneous equation. We have the following formulas for c_1 and c_2:

$$\begin{aligned}
c_1(t) &= -\int \frac{y_2 f}{W} \mathrm{d}t \\
c_2(t) &= \int \frac{y_1 f}{W} \mathrm{d}t
\end{aligned}$$

where $W(t) = y_1(t) y_2'(t) - y_1'(t) y_2(t)$ is the Wronskian determinant.

We compute

$$\begin{aligned}
y_1 &= e^{-2t} \\
y_1' &= -2e^{-2t}
\end{aligned}$$

$$\begin{aligned} y_2 &= te^{-2t} \\ y_2' &= (-2t+1)e^{-2t} \end{aligned}$$

$$W(t) = y_1 y_2' - y_1' y_2 = (-2t+1)e^{-4t} + 2te^{-4t} = e^{-4t}$$

Now we use our formulas to find c_1 and c_2.

$$\begin{aligned} c_1(t) &= -\int \frac{y_2 f}{W}\mathrm{d}t \\ &= -\int \frac{te^{-2t}(\frac{1}{t^2}e^{-2t})}{e^{-4t}}\mathrm{d}t \\ &= -\int \frac{1}{t}\mathrm{d}t \\ &= -\ln t \end{aligned}$$

$$\begin{aligned} c_2(t) &= \int \frac{y_1 f}{W}\mathrm{d}t \\ &= \int \frac{e^{-2t}(\frac{1}{t^2}e^{-2t})}{e^{-4t}}\mathrm{d}t \\ &= \int \frac{1}{t^2}\mathrm{d}t \\ &= -\frac{1}{t} \end{aligned}$$

Then $y_p = -e^{-2t}\ln t - e^{-2t}$ is a particular solution. So, the general solution is $y = -e^{-2t}\ln t + k_1 e^{-2t} + k_2 te^{-2t}$.

Problem 4. A mass of 100 grams stretches a spring 3 cm (this determines the spring constant k, since restoring force equals k times displacement by Hooke's Law; be aware that force is mass times gravitational constant; make sure you get the units right). If the mass is set in motion from its equilibrium position with a downwards velocity of 10 cm/sec, and if there is no damping, determine the position of the mass at any time t. When does the mass first return to its equilibrium position?

The differential equation is $my'' + \gamma y' + ky = F(t)$ where m is the mass, γ is the damping constant, k is the spring constant, F is the external force and y is the displacement of the mass from its equilibrium position. We measure the downward direction as positive.

Then $m = 100$ g $= 0.1$ kg, $\gamma = 0$ and $F(t) = 0$. To find the spring constant, we use Hooke's law, which gives the relation $mg = kL$, where $g = 9.81$ m/s^2 is gravity and $L = 3$ cm $= 0.03$ m is the elongation of the spring by the mass. So, $k = (0.1)(9.81)/0.03 = 32.7$. The equation becomes

$$0.1y'' + 32.7y = 0$$

This is a second order linear ODE with constant coefficients and the general solution is $y = k_1\cos(\sqrt{327}t) + k_2\sin(\sqrt{327}t)$. To find the constants k_1 and k_2 we need a pair of initial conditions. Since the mass begins at its equilibrium position, $y(0) = 0$. And since the initial velocity is 10 cm/s $= 0.1$ m/s downwards, $y'(0) = 0.1$. Solving this IVP, we get $y(t) = \frac{\sqrt{327}}{3270}\sin(\sqrt{327}t)$ is the position of the mass at any time t. The mass returns to the equilibrium position when $y(t) = 0$. Setting $\frac{\sqrt{327}}{3270}\sin(\sqrt{327}t) = 0$, we must have $\sin(\sqrt{327}t) = 0$, or $t = \frac{n\pi}{\sqrt{327}}$ for some integer n. Thus, the mass first returns to equilibrium after $\frac{\pi}{\sqrt{327}} \approx 0.17$ seconds.

Problem 5. Consider the forced but undamped system described by the initial value problem

$$y'' + y = 3\cos(\omega t)\,, \quad y(0) = y'(0) = 0\,.$$

(i) Find the solution $y(t)$ for $\omega \neq 1$.

First we solve the homogeneous equation $y_h'' + y_h = 0$ and get the fundamental set of solutions $y_1 = \cos t$ and $y_2 = \sin t$. To find a particular solution, we guess $y_p = A\cos(\omega t) + B\sin(\omega t)$. Solving for A and B, we find $y_p = \frac{3}{1-\omega^2}\cos(\omega t)$ is one solution. Then the general solution is $y = \frac{3}{1-\omega^2}\cos(\omega t) + k_1\cos t + k_2\sin t$. Using the initial conditions to find the constants, the desired solution is $y = \frac{3}{1-\omega^2}\cos(\omega t) + \frac{3}{\omega^2-1}\cos t$.

(ii) Plot the solution $y(t)$ versus t for $\omega = 0.7$, $\omega = 0.8$, and $\omega = 0.9$. Describe how $y(t)$ changes as ω varies in this interval. What happens as ω takes on values closer and closer to 1? Note that the natural frequency of the unforced system is $\omega_0 = 1$.

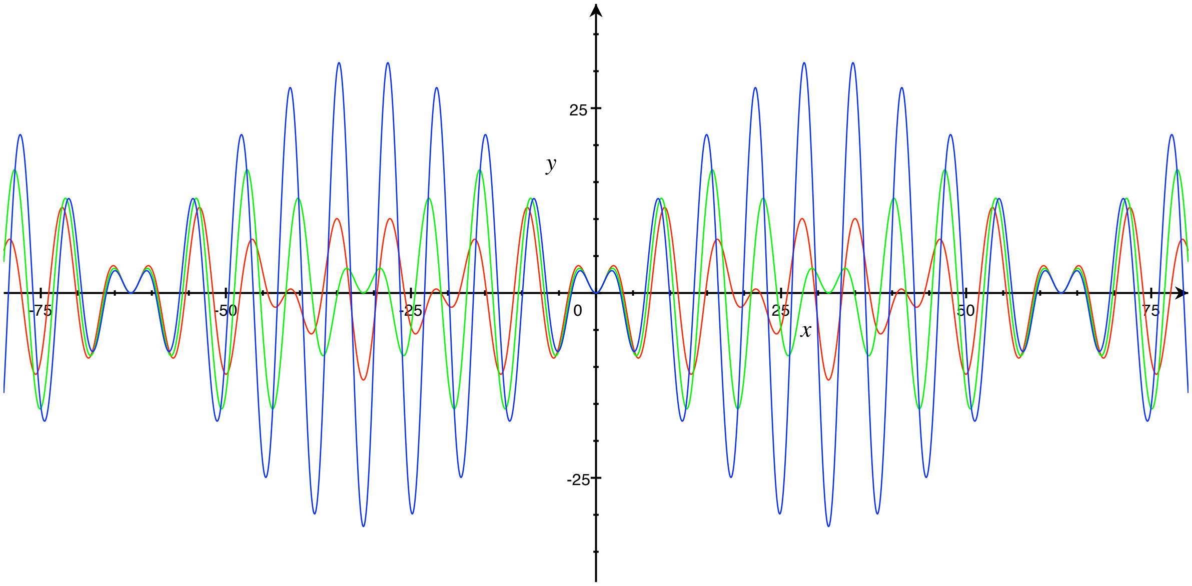

Figure 1. red: $\omega = 0.7$, green: $\omega = 0.8$, blue: $\omega = 0.9$