

HOMEWORK 9, M 331  
DUE 4/16/09

**Problem 1.** Find the general solution of the inhomogeneous ODE

$$y^{(4)} - y = 3t + \sin t.$$

First we solve the corresponding homogeneous ODE  $y_h^{(4)} - y_h = 0$ . Looking for a solution of the form  $y_h = e^{\lambda t}$  we see that  $\lambda$  must satisfy

$$\lambda^4 - 1 = 0.$$

Factoring,

$$\begin{aligned}(\lambda^2 - 1)(\lambda^2 + 1) &= 0 \\(\lambda - 1)(\lambda + 1)(\lambda - i)(\lambda + i) &= 0 \\ \lambda &= \pm 1, \pm i\end{aligned}$$

From these roots we obtain the four fundamental solutions:  $y_1 = e^t$ ,  $y_2 = e^{-t}$ ,  $y_3 = \cos t$  and  $y_4 = \sin t$ . Thus, the general solution of the homogeneous ODE is  $y_h = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t$ .

The general solution of the inhomogeneous ODE is  $y = y_p + y_h$  where  $y_p$  is any particular solution. So, we need to find  $y_p$ .

The right hand side is the sum of the two terms  $3t$  and  $\sin t$ . Since the ODE is linear, we can consider separately the ODEs

$$u^{(4)} - u = 3t \quad \text{and} \quad v^{(4)} - v = \sin t.$$

If  $u$  and  $v$  solve the above equations, by linearity the sum  $y_p = u + v$  will solve the equation  $y_p^{(4)} - y_p = 3t + \sin t$ . Since the right hand side of the equation for  $u$  is  $3t$ , we guess a solution of the form  $u = At + B$ .

$$\begin{aligned}u &= At + B \\ u' &= A \\ u'' &= u^{(3)} = u^{(4)} = 0\end{aligned}$$

Substituting this into the equation  $u^{(4)} - u = 3t$ , we have

$$-At - B = 3t$$

Equating coefficients,

$$\begin{aligned}-A &= 3 && \text{from the } t \text{ term} \\ B &= 0 && \text{from the constant term}\end{aligned}$$

Then  $A = -3$  and  $B = 0$ , so  $u = -3t$ .

Now we guess a solution to the second equation  $v^{(4)} - v = \sin t$ . Since the right hand side is  $\sin t$ , this suggests we guess a solution of the form  $v = C \cos t + D \sin t$ . However, we have determined above that this is a solution to the homogenous equation. So, instead we multiply by  $t$  and guess a solution of the form  $v = t(C \cos t + D \sin t)$ .

$$\begin{aligned} v &= t(C \cos t + D \sin t) \\ v' &= (Dt + C) \cos t + (-Ct + D) \sin t \\ v'' &= (-Ct + 2D) \cos t + (-Dt - 2C) \sin t \\ v^{(3)} &= (-Dt - 3C) \cos t + (Ct - 3D) \sin t \\ v^{(4)} &= (Ct - 4D) \cos t + (Dt + 4C) \sin t \end{aligned}$$

Substituting this into the equation  $v^{(4)} - v = \sin t$ , we have

$$(Ct - 4D) \cos t + (Dt + 4C) \sin t - (Ct \cos t + Dt \sin t) = \sin t$$

or

$$-4D \cos t + 4C \sin t = \sin t$$

Equating coefficients,

$$\begin{aligned} -4D &= 0 && \text{from the } \cos t \text{ term} \\ 4C &= 1 && \text{from the } \sin t \text{ term} \end{aligned}$$

Then  $D = 0$  and  $C = 1/4$ , so  $v = \frac{1}{4}t \cos t$ . Since  $y_p = u + v$ , we have  $y_p = -3t + \frac{t}{4} \cos t$ . Therefore, the general solution is  $y = -3t + \frac{t}{4} \cos t + c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t$ .

**Problem 2.** Find the solution of the inhomogeneous ODE

$$y^{(3)} + 4y' = -t$$

with initial conditions  $y(0) = y'(0) = 0$ ,  $y''(0) = 1$ .

First we solve the corresponding homogeneous ODE  $y_h^{(3)} + 4y_h' = 0$ . Looking for a solution of the form  $y_h = e^{\lambda t}$  we see that  $\lambda$  must satisfy

$$\lambda^3 + 4\lambda = 0.$$

Factoring,

$$\begin{aligned} \lambda(\lambda^2 + 4) &= 0 \\ \lambda(\lambda - 2i)(\lambda + 2i) &= 0 \\ \lambda &= 0, \pm 2i \end{aligned}$$

From these roots we obtain the three fundamental solutions:  $y_1 = 1$ ,  $y_2 = \cos 2t$  and  $y_3 = \sin 2t$ . Thus, the general solution of the homogeneous ODE is  $y_h = c_1 + c_2 \cos 2t + c_3 \sin 2t$ .

The general solution of the inhomogeneous ODE is  $y = y_p + y_h$  where  $y_p$  is any particular solution. So, we need to find  $y_p$ .

The right hand side is  $-t$ , which suggests we guess a polynomial. Since the terms on the left hand side are  $y^{(3)}$  and  $y'$ , we need at least a second order polynomial to be left with a  $t$  term after differentiating. So, we guess  $y_p = At^2 + Bt$ , omitting the constant term since we saw above that constants solve the homogeneous equation.

$$\begin{aligned}y_p &= At^2 + Bt \\y_p' &= 2At + B \\y_p'' &= 2A \\y_p^{(3)} &= 0\end{aligned}$$

Substituting this into the equation, we have

$$4(2At + B) = -t$$

or

$$8At + 4B = -t$$

Equating coefficients,

$$\begin{aligned}8A &= -1 && \text{from the } t \text{ term} \\4B &= 0 && \text{from the constant term}\end{aligned}$$

Then  $A = -1/8$  and  $B = 0$ , so  $y_p = -t^2/8$ . Therefore, the general solution is  $y = -\frac{1}{8}t^2 + c_1 + c_2 \cos 2t + c_3 \sin 2t$ . The solution to the initial value problem is  $y = -\frac{1}{8}t^2 + \frac{5}{4} - \frac{5}{4} \cos t$ .

**Problem 3.** Find the general solution of the inhomogeneous ODE

$$y^{(3)} + y'' + y' + y = e^{-t}.$$

First we solve the corresponding homogeneous ODE  $y_h^{(3)} + y_h'' + y_h' + y_h = 0$ . Looking for a solution of the form  $y_h = e^{\lambda t}$  we see that  $\lambda$  must satisfy

$$\lambda^3 + \lambda^2 + \lambda + 1 = 0.$$

Factoring,

$$\begin{aligned}\lambda^2(\lambda + 1) + (\lambda + 1) &= 0 \\(\lambda + 1)(\lambda^2 + 1) &= 0 \\(\lambda + 1)(\lambda + i)(\lambda - i) &= 0 \\\lambda &= -1, \pm i\end{aligned}$$

From these roots we obtain the three fundamental solutions:  $y_1 = e^{-t}$ ,  $y_2 = \cos t$  and  $y_3 = \sin t$ . Thus, the general solution of the homogeneous ODE is  $y_h = c_1 e^{-t} + c_2 \cos t + c_3 \sin t$ .

The general solution of the inhomogeneous ODE is  $y = y_p + y_h$  where  $y_p$  is any particular solution. So, we need to find  $y_p$ .

The right hand side is  $e^{-t}$ , which suggests we guess  $y_p = Ae^{-t}$ . However, this is a solution to the homogeneous ODE, so instead we multiply by  $t$  and guess  $y_p = Ate^{-t}$ .

$$\begin{aligned}y_p &= Ate^{-t} \\y_p' &= A(1-t)e^{-t} \\y_p'' &= A(t-2)e^{-t} \\y_p^{(3)} &= A(3-t)e^{-t}\end{aligned}$$

Substituting this into the equation, we have

$$A(3-t)e^{-t} + A(t-2)e^{-t} + A(1-t)e^{-t} + Ate^{-t} = e^{-t}$$

or

$$2Ae^{-t} = e^{-t}$$

Then  $A = 1/2$  and  $y_p = \frac{t}{2}e^{-t}$ . Therefore, the general solution is  $y = \frac{t}{2}e^{-t} + c_1e^{-t} + c_2 \cos t + c_3 \sin t$ .