Practice Midterm Problems Math 235, Fall 2011

1. a: Write the system of equations below as a matrix equation:

$$y - z = 1$$

$$x + 2y + 2 = -2$$

$$-x - 2y - z = 3$$

   b: Solve the system of equations using row operations.

2. a: For what vectors $v = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ does the equation $Ax = v$ have a solution if $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 3 \end{pmatrix}$, and $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$.
   b: What is the rank of the matrix A?
   c: What is the dimension of $\mathrm{im}(A)$? What is the dimension of $\ker(A)$?

3. What does it mean for a vector to be in the image of a matrix A.
   Let A be the matrix $\begin{pmatrix} 1 & 2 & 5 \\ -2 & 0 & -2 \\ 3 & -1 & 1 \end{pmatrix}$. Is $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ an element of the image of A? Why or why not?

4. a: Define what it means for a set S to be a basis of a subspace $V \subset \mathbb{R}^n$.
   b: Let

$$A = \begin{pmatrix} 1 & 2 & 3 & -1 \\ -1 & 0 & 1 & -1 \\ -1 & 4 & 3 & -5 \end{pmatrix}.$$

   Give a set of vectors that span $\mathrm{im}(A)$ and that are independent.

5. a: Let A be a $p \times q$ matrix, so A gives a linear map from $\mathbb{R}^q$ to $\mathbb{R}^p$. Let $X_1, X_2 \in \mathbb{R}^q$. Assume that $A(X_1) = B$ and that $A(X_2) = 0$. Explain why $A(X_1 + X_2) = B$.

   b: Let X_1 be a solution to $A(X) = B$. Explain why every solution to $A(X) = B$ can be written as $X = X_1 + X_2$ with X_2 a solution to $AX = 0$.

6. Let A be a two by two matrix that rotates by angle $\pi/4$.
   a: Find A.
   b: Give a geometric explanation for why $A^{24} = I_2$. (Here I_2 denotes the 2 by 2 identity matrix.) What other powers of A yield I_2? What powers of A yield $-I_2$?

7. Solve the equation

$$\begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 2 \\ 2 & 1 & -1 \end{pmatrix} X = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}.$$

for $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ by first finding the inverse of the given coefficient matrix.

8. Compute the products AB and BA of the two matrices A, B given below, and compare their ranks:

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 0 \\ 3 & -2 \end{pmatrix}$$

$$B = \begin{pmatrix} -1 & 0 & 0 \\ 4 & 8 & 7 \end{pmatrix}$$

9. Let W denote the set of vectors in $\mathbb{R}^3$ orthogonal to the vector $\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$. Argue that W is a subspace and find a basis of W.

10. Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 2 & 0 \\ 1 & 1 & 3 \\ -2 & 1 & -3 \end{pmatrix}.$$

Let $B = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$. Find equations in b_1, b_2, b_3, b_4 so that the equation $AX = B$ can be solved. Find a basis of the image of A.

11. True or False. (Please give a reason if True or a counterexample if False.)
a: The image of a 3×4 matrix A is a subspace of $\mathbb{R}^4$.

b: If the kernel of a matrix A consists of the zero vector only, then the column vectors of A must be linearly independent.

c: If u, v, w are in a subspace of any vector space, then so is $3u - v - w$.

d: The function

$$T : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x - y \\ x + y \end{pmatrix}$$

is a linear map.

e: If A is an invertible $n \times n$ matrix, then the reduced row echelon form of A is I_n.

f: The formula $AB = BA$ holds for all $n \times n$ matrices A and B.

g: If the 4×4 matrix A has rank 4, then any linear system with coefficient matrix A will have a unique solution.

12. An $n \times n$ matrix M defines a linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$. Note that M^2 also defines a linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$.
a: Show that if $v \in \text{im}(M^2)$, then $v \in \text{im}(M)$, meaning we have the inclusion of subspaces $\text{im}(M^2) \subset \text{im}(M)$.
b: What are the corresponding statements about $\ker(M)$ and $\ker(M^2)$?
c: Find a 3×3 matrix M with $\text{im}(M) \neq \text{im}(M^2)$ and also (necessarily – why?!) $\ker(M) \neq \ker(M^2)$.

13. Let $f : \mathbb{R}^3 \to \mathbb{R}^5$ be a linear map. We are given that

$$f(u) = (0, 1, 2, 3, 4)$$
$$f(v) = (-1, 2, 6, 1, 4).$$

What is $f(2u - 3v)$?

14. Which of the following are subspaces of the indicated space. Explain your answer.
a: The set of of solutions in $\mathbb{R}^3$ to the equation $3x - y + 2z = 1$.
b: The set of vectors in $\mathbb{R}^4$ orthogonal to $(1, 2, 3, -4)$.
c: The subset of all 3×3 matrices A satisfying $A^T = 2A$.

15. The matrix M of size 4×6 has kernel with dimension 2. How many independent column vectors does M have? Why? What is the dimension of the image of M? Why ?

16. Let

$$A = \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}, B = \begin{pmatrix} -1 & -4 \\ 4 & 7 \end{pmatrix}.$$

Find a 2×2 matrix X so that

$$AX = B.$$

17. The set

$$B = \left\{ \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$$

is a basis of $\mathbb{R}^2$.

a: Find the coordinates with respect to the standard basis of the vector v whose coordinates with respect to B are

$$[v]_B = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

b: Find the coordinates with respect to the basis B of the vector with coordinates

$$\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

in the standard basis.

18. Let

$$M = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

be the matrix of a linear transformation with respect to the standard basis. Let

$$B = \{ \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \}$$

be a basis of $\mathbb{R}^2$. What is the matrix $[M]_B$ of the linear transformation of M with respect to the basis B.

19. For each of the linear maps below find a basis of $\mathbb{R}^2$ so that its matrix with respect to the basis is diagonal.

a: Orthogonal projection on the line in $\mathbb{R}^2$ spanned by $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

b: Reflection across the line $y = 2x$.