Math 235 Final Exam Review Materials

(courtesy of Pat “the magic” Dragon)

This summary is meant as a study guide. This is not a comprehensive resource, and will not singlehandedly prepare students for the final exam. But it could streamline things a bit.

Please enjoy the following *Inspirational Content*: `https://www.youtube.com/watch?v=3ieaDfD_h6s`. (Even the course chair recommends these historical short videos!)

1 Definitions

The following definitions must be memorized:

- An Eigenvector for a matrix
- An Eigenvalue for a matrix
- An Eigenspace for a matrix
- The characteristic equation for a matrix
- The multiplicity of an Eigenvalue
- Similar matrices
- A diagonal matrix
- A diagonalizable matrix
- The dot product of two vectors
- The norm of a vector
- Orthogonal vectors
- Orthogonal sets of vectors
- Orthonormal sets of vectors
- Orthogonal projection of a vector

2 Key Topics

This is a very brief reminder of important topics and skills we've discussed and practiced this semester.

2.1 Solving Eigenvector Problems

- Be able to find the Eigenvalues of a square matrix, and know how to identify their multiplicities.
- For each Eigenvalue, be able to find a basis of Eigenvectors for the corresponding Eigenspace.
- Know how the eigenvalues of similar matrices are related.
- Be familiar with the invertible matrix theorem and how it relates to Eigenvalues.

2.2 Diagonalization

- Be able to determine whether or not a matrix is diagonalizable.
- Know how to find matrices C and D so that $X = CDC^{-1}$.
- Be able to use this to calculate high powers of a matrix.

2.3 Orthogonality

- Know how to calculate the dot product of two vectors.
- Be able to determine whether or not a pair of vectors is orthogonal.
- Be able to determine whether or not a set of vectors is orthogonal.
- Know how to decompose a vector in terms of orthogonal projections.
- Be able to determine whether or not a set of vectors is orthonormal.

3 Suggested Exercises

The following exercises should provide some useful review for the upcoming midterm exam. Again, this is by no means comprehensive.

3.1 Examples

For each of the following, provide an example. As always, you need to fully justify your claim.

(a) A matrix with three distinct Eigenvalues

(b) A 3×3 matrix with exactly two distinct Eigenvalues

(c) A 2×2 matrix with only one linearly independent Eigenvector

(d) A matrix with no real Eigenvalues

(e) Similar matrices

(f) Matrices which have the same Eigenvalues but are not similar

(g) A matrix with Eigenvalues 3 and 4, occurring at multiplicities 5 and 6, respectively.

(h) A diagonal matrix

(i) A diagonalizable matrix

(j) A matrix which is not diagonalizable

(k) A vector with length three.

(l) An orthonormal set of four vectors.

3.2 Eigenvalue Problems

Let $M = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$. Find the Eigenvalues and Eigenvectors for M. Use these to find matrices C and D such that $M = CDC^{-1}$ and D is diagonal. (Some folks, especially the course chair, use a capital Lambda — Λ — to denote a diagonal matrix. And the course chair likes to remember the eigen-equation as $MC = C\Lambda$.)

3.3 Similar Matrices

Suppose that A and B are $n \times n$ matrices, and that A is similar to B. Show that A and B have the same characteristic polynomial.

3.4 Impossible Diagonalization

Show that the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ is not diagonalizable.

3.5 Dot Products to the Rescue

Suppose that $ABCD$ is a parallelogram. Prove that if $ABCD$ is a rhombus, then its diagonals are orthogonal.

3.6 Taking Powers of a Matrix

(a) Provide an example of a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $A^2 \neq \begin{pmatrix} a^2 & b^2 \\ c^2 & d^2 \end{pmatrix}$.

(b) Let $D = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$. Calculate D^5, and deduce a general formula for D^k.

(c) Let A be a 2×2 matrix. Suppose that $A = CDC^{-1}$, for some invertible matrix C. Let k be a positive integer. Show that $A^k = CD^kC^{-1}$.

(d) Let $A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$. Calculate A^{100}.

3.7 Constructing Orthonormal Bases

Let $\vec{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, and $\vec{w} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$.

(a) Verify that $\{\vec{u}, \vec{v}, \vec{w}\}$ is a linearly independent set, but not an orthogonal set.

(b) Let $\vec{a} = \vec{u}$, let $\vec{b} = \vec{v} - \text{ proj}_{\vec{a}}(\vec{v})$, and let $\vec{c} = \vec{w} - \text{ proj}_{\vec{a}}(\vec{w}) - \text{ proj}_{\vec{b}}(\vec{w})$. Show that $\{\vec{a}, \vec{b}, \vec{c}\}$ is an orthogonal set.

(c) Use this to construct an orthonormal basis for $\mathbb{R}^3$.

3.8 Keep Beating the Drum

Consider the following vectors in $\mathbb{R}^\infty$:

$$\vec{u} = \begin{pmatrix} 1 \\ \frac{1}{2} \\ \left(\frac{1}{2}\right)^2 \\ \left(\frac{1}{2}\right)^3 \\ \vdots \end{pmatrix}$$

$$\vec{v} = \begin{pmatrix} 1 \\ \frac{1}{3} \\ \left(\frac{1}{3}\right)^2 \\ \left(\frac{1}{3}\right)^3 \\ \vdots \end{pmatrix}$$

(a) Find the angle between $\vec{u}$ and $\vec{v}$.

(b) Find a vector $\vec{w}$ so that $\vec{w} \perp \vec{v}$.

(c) Find a vector $\vec{w}$ so that the angle between $\vec{u}$ and $\vec{w}$ is $\frac{\pi}{3}$.

(d) Is it possible to find a vector $\vec{w}$ satisfying both of these properties (b,c)?