Math 235 Fall 2019 Midterm 1 Review Ideas

This summary (prepared by Pat Dragon) is meant as a study guide. This is not a comprehensive resource, and will not singlehandedly prepare students for the midterm exam. But it should streamline things a bit.

1 Definitions

The following definitions must be memorized:

- Matrices, rows, and columns
- Row echelon form
- Reduced row echelon form
- Pivots, pivot positions, pivot columns
- Consistent system of equations, inconsistent system of equations
- Parametric vector form
- Linear combination, trivial linear combination, nontrivial linear combination
- Span of a set of vectors, whether a set of vectors spans a space
- Linearly independent, linearly dependent
- Linear map (or linear transformation)
- Identity matrix
- Standard basis vectors
- Matrix transpose
- Invertible matrix
- Matrix inverse

2 Key Topics

This is a very brief reminder of important topics and skills we've discussed and practiced this semester.

2.1 Putting Matrices in (R)REF

- Be able to dentify if a matrix is in row echelon form (REF), or in reduced row echelon form (RREF)
- Know how to take a given matrix and put it into RREF using row operations.

2.2 Solving Linear Systems

See the *Existence and Uniqueness Theorem*, on page 21 of your text. It basically tells us how to use the RREF of an augmented matrix to predict what sorts of solutions the system will have.

- If the RREF gives us a bottom row corresponding to $0 = 1$, then the system is inconsistent.
- If the RREF gives us no free variables, then the system has a unique solution.
- If the RREF gives us at least one free variable, then the system has infinitely many solutions.

We can also use the RREF to solve the system, as in the following section.

2.2.1 Parametric Vector Form

Be able to solve a linear system of a equations, and write the solution in parameterized vector form.

2.3 Linear Combinations

We need to know what linear combinations are, and how they can be used to build new vectors from some given vectors.

2.3.1 Spans

- Be able to determine if a set of vectors span $\mathbb{R}^n$.
- Be able to determine if a given vector $\vec{u}$ is in the span of another given set of vectors.
- Know how theorem 4 (page 37) relates to solving linear systems.

2.3.2 Linear Independence

Be able to determine whether or not a given set is linearly independent. Some key cases:

- If there are more vectors than components, then your set is dependent.
- If one of the vectors is $\vec{0}$, then your set is dependent.
- If one of your vectors is a linear combination of the others, then your set is dependent.
- If n vectors span $\mathbb{R}^n$, then they're independent.

2.4 Linear Maps or Transformations

Be able to determine whether or not a given map or transformation is linear, and interpret formulas for them from geometric instructions.

2.4.1 Matrix for a Linear Map (or Transformation)

Be able to write the (standard) matrix for a given linear map. Recall, the jth column is given by using $\vec{e}_j$ as an input.

2.4.2 Standard Basis Vectors

Know what they are, and how they're used to get the columns of matrices.

2.4.3 Identity Matrices

Know what the identity matrix is, and how it's related to the inverse of a matrix.

2.5 Operations with Matrices

- Addition, subtraction, and multiplication of matrices.
- Note that in general, $AB \neq BA$.
- Given a matrix X, its *transpose*, denoted X^T, simply replaces rows with columns, and vice versa.
- $(XY)^T = Y^T X^T$

2.5.1 Matrix Inverses

- Know how to check if a 2×2 matrix is invertible, and calculate its inverse.
- Know how to check if a larger square matrix is invertible, and calculate its inverse.
- Be familiar with the invertible function theorem (page 114).

3 Suggested Exercises

The following exercises should provide some useful review for the upcoming midterm exam. Again, this by no means comprehensive.

3.1 Examples

For each of the following, provide an example. As always, you need to fully justify your claim.

(a) A matrix with 7 rows and 5 columns.

(b) A matrix which is not in row echelon form or reduced row echelon form

(c) A matrix in row echelon form, but not reduced row echelon form

(d) A matrix in reduced row echelon form

(e) A matrix with six columns, and pivots in columns 1,4,5

(f) A matrix with a pivot in every column

(g) An inconsistent system of equations

(h) A system of equations with exactly one solution

(i) A system of equations with infinitely many solutions

(j) A solution to a linear system of equations, written in parametric vector form

(k) A nontrivial linear combination of the vectors $\vec{u}, \vec{v}, \vec{w}$

(l) A vector in span$\{\vec{u}, \vec{v}\}$

(m) A linearly independent set of three vectors

(n) A set of three vectors $\{\vec{u}, \vec{v}, \vec{w}\}$ which is linearly independent, but $\{\vec{u}, \vec{v}\}$, $\{\vec{u}, \vec{w}\}$, and $\{\vec{w}, \vec{v}\}$ are linearly dependent

(o) A linear map or transformation

(p) A map or transformation $\mathbb{R}^4\mathbb{R}^3$ which is not linear

(q) The identity matrices I_3 and I_4

(r) In $\mathbb{R}^5$, the standard basis vectors

(s) A nonzero 3×3 matrix X (not all of the entries of X can be zero) such that $X^T = -X$

(t) A 2×2 matrix Y such that $Y^T = Y^{-1}$

3.2 Augmented Matrices

Write the augmented matrix for the following system of equations:

$$\begin{aligned} x_1 - 3x_2 + x_4 &= 1 \\ x_1 + x_2 - x_5 &= 0 \\ -2x_1 + x_4 &= 1 \end{aligned}$$

3.3 Basic Matrix Arithmetic

Let A be the augmented matrix described above. Calculate A^TA and AA^T.

3.4 RREF

Put A in reduced row echelon form.

3.5 Solving Systems

Using the RREF for A, identify the free variables, and solve for each dependent variable in terms of the free variables. Then write the solution to the system in parameterized vector form.

3.6 Linear Independence

Let $S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$. Determine whether or not this is a linearly independent set.

3.7 Linear Maps (or Transformations)

Consider the map $\mathbb{R}^2 \xrightarrow{T} \mathbb{R}^2$ which reflects every point across the line $y = -x$. Show that T is a linear map by explicitly verifying the defining properties of a linear map , and find a matrix for T.

Consider the map $\mathbb{R}^2 \xrightarrow{U} \mathbb{R}^2$ which reflects every point across the line $y = x + 1$. Show that U is not a linear map .

Consider the map $\mathbb{R}^2 \xrightarrow{V} \mathbb{R}^2$ which rotates every point about the point $(1, 1)$ by an angle of π. Determine whether or not V is a linear map.

3.8 More Matrix Arithmetic

Let $X = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 2 & 2 \end{pmatrix}$, and let $Y = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

(a) Calculate X^{-1}.

(b) Find a matrix Z such that $X^{-1}ZX = Y$.

3.9 Shenanigans with Definitions

Let $\mathbb{R}^n \xrightarrow{S} \mathbb{R}^m$ be a linear transformation, and let $\vec{u}, \vec{v}, \vec{w}$ be vectors in $\mathbb{R}^n$. Suppose that $\vec{x}$ is a vector in $\operatorname{span}\{\vec{u}, \vec{v}, \vec{w}\}$. Prove that $S(\vec{x})$ is in $\operatorname{span}\{S(\vec{u}), S(\vec{v}), S(\vec{w})\}$