Math 235 Midterm 2 Review Materials

(courtesy of Dr. Pat Dragon)

This summary is meant as a study guide. This is not a comprehensive resource, and will not singlehandedly prepare students for the midterm exam. But it should streamline things a bit.

Please enjoy the following *Inspirational Content*: `https://www.youtube.com/watch?v=lIES3ii-IOg`.

1 Definitions

The following definitions must be memorized:

- Vector space
- Subspace of a vector space
- Null space (or Kernel)
- Column space (or Image)
- Linear transformation between vector spaces
- Linearly independent set in a vector space
- Basis of a vector space
- Dimension of a vector space
- Rank of a matrix

2 Key Topics

This is a very brief reminder of important topics and skills we've discussed and practiced this semester.

2.1 Calculating Determinants

- Memorize the formula for the determinant of a 2×2 matrix.
- Know how to calculate the determinant of a larger matrix using cofactors.

2.2 Properties of Determinants

- Know how each kind of row operation affects the determinant of a matrix.
- Know how determinants are related to invertibility of matrices.
- Be familiar with the determinant's nice algebraic properties: $\det(AB) = \det(A)\det(B)$, $\det(A^T) = \det(A)$, $\det(I) = 1$, $\det(0) = 0$, etc.
- Know how determinants can be used to calculate the areas of parallelograms, and volumes of parallelepipeds.
- Know how the determinant is interpreted geometrically: the absolute value of the determinant is the ratio by which areas (in 2 dimensions) or volumes (in 3 dimensions) are stretched.

2.3 Cramer's Rule

- Be able to use Cramer's rule to solve linear systems.
- Be able to use Cramer's rule to calculate the inverse of a matrix.

2.4 Vector Spaces

- Know lots of examples of vector spaces.
- Be able to check whether or not a nonempty set is a vector space.
- Be able to prove basic properties, like $0\vec{v} = \vec{0}$, $\alpha\vec{0} = \vec{0}$, and $-\vec{u} = (-1)\vec{u}$.
- Know lots of examples of subspaces.
- Be able to check whether or not a subset is a subspace.

- Be able to determine whether or not a function between vector spaces is a linear transformation.

2.5 Null Space (Kernel) and Column Space (Image)

- Be able to determine the null space (or kernel) of a matrix.
- Be able to determine the column space (or image) of a matrix.
- Know that if A is an $m \times n$ matrix, then $\mathrm{Col}(A)$ (or image) is a subspace of $\mathbb{R}^m$.
- Know that if A is an $m \times n$ matrix, then $\mathrm{Nul}(A)$ (or kernel) is a subspace of $\mathbb{R}^n$.

2.6 Linear Independence and Bases

- Be able to check whether or not a set of vectors in a vector space is linearly independent.
- Be able to check whether or not a set of vectors spans a subspace.
- Be able to check whether or not a set of vectors is a basis for a subspace.
- Be able to construct bases for the null space and column space (or kernel and image) of a matrix.

2.7 Coordinates

- Know how to calculate the coordinates and coordinate vector for a given vector, with respect to a given basis.
- Know how to calculate a change of basis matrix.

2.8 Dimension and Rank

- Be able to calculate the dimension of a subspace.
- Know how the dimension of a vector space and the dimension of a subspace are related.

- Know how to calculate the rank (dimension of image) of a matrix.
- Know that for an $m \times n$ matrix A, the rank of A, plus the nullity of A (the dimension of its null space or kernel), equals n.
- Be familiar with the incarnation of the invertible matrix theorem on page 237.

3 Suggested Exercises

The following exercises should provide some useful review for the upcoming midterm exam. Again, this by no means comprehensive.

3.1 Examples

For each of the following, provide an example. As always, you need to fully justify your claim.

(a) A matrix with determinant zero

(b) A matrix with nonzero determinant

(c) A row operation which changes the determinant

(d) A row operation which does not change the determinant

(e) A vector space

(f) A set which is not a vector space

(g) A subspace

(h) A subset which is not a subspace

(i) A linear transformation between vector spaces that aren't $\mathbb{R}^q$

(j) A function between vectors spaces that aren't $\mathbb{R}^q$, which is not a linear transformation

(k) The null space (kernel) of a matrix

(l) A basis for the null space (or kernel) of a matrix

(m) The column space (or image) of a matrix

(n) A basis for the null space (or kernel) of a matrix

(o) A linearly independent set in a vector space that isn't $\mathbb{R}^q$

(p) A linearly dependent set in a vector space

(q) Coordinates for a vector with respect to a given basis

3.2 Determinants

Let $M = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}$. Calculate $\det(M)$.

3.3 Properties of Determinants

Let A be a 3×3 matrix such that $A^T = -A$. What are the possible values of $\det(A)$?

3.4 Areas and Determinants

Use a determinant to calculate the area of the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$. Hint: a triangle is half of a parallelogram.

3.5 Cramer's Rule

Let $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}$. Let $\vec{b} = \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix}$.

(a) Use Cramer's rule to solve $A\vec{x} = \vec{b}$.

(b) Use Cramer's rule to calculate A^{-1}.

3.6 Subspaces

Let V be the set of 3×3 matrices, let H be the set of 3×3 matrices with at least one column of all zeroes, and let K be the set of 3×3 matrices whose third column is all zeroes. Show that H is not a subspace of V, but K is.

3.7 Null Space (Kernel) and Column Space (Image)

Let $A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$. Find bases for $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$, and determine the dimensions of these subspaces.

3.8 Bases of Vector Spaces

Let V be the set of 2×2 matrices, and let H be the set of 2×2 matrices of the form $\begin{pmatrix} a & a \\ b & -b \end{pmatrix}$, where a and b are arbitrary real numbers. Let

$$B = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \right\}, C = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} \right\}$$

Which of these is a basis for H?

3.9 Coordinates

Let V be the set of polynomials, and let H be the set of polynomials of the form $p(x) = ax^3 + ax^2 + bx + c$, where a, b, c are arbitrary real numbers. Let $p(x) = x^3 + x^2 + x + 1$, let $q(x) = 2x + 1$, and let $r(x) = x - 1$. Let $B = \{p(x), q(x), r(x)\}$.

For this exercise, you may assume that H is a subspace of V, and that B is a basis for H. Let $s(x) = x^2 + 1$. Find the coordinate vector for $s(x)$ with respect for B.

3.10 End Boss: The Final Form

Let V be the set of smooth functions (functions which are infinitely many times differentiable), and let $V \xrightarrow{L} V$ be the linear operator defined by

$$L(f) = \frac{d^2 f}{dx^2}$$

For this question, you may assume that V is a vector space, and that L is a linear transformation. Determine the null space (or kernel) for L, find a basis for the kernel, and determine its dimension. [I approve this message — *The Boss RK* ;-]