Section 1 - Matrix Algebra and Row Reduction

- Elementary Row Operations (ERO's):
  1. Add a row to another row
  2. Multiply a row by some real number
  3. Swap 2 rows.

These ERO's preserve the solution of a matrix.

- REF & RREF? More like WTF & WWTF!

Actually, no. More like:

A matrix A is in REF if:
1. All $\vec{0}$ rows are at the bottom
2. Every row's leading nonzero entry is to the right of the row above it

A Matrix A in RREF satisfies 1 & 2 and
3. The leading nonzero entry in each row is 1
4. If a column contains a leading 1, all other entries in that column are 0

Every matrix has exactly 1 RREF

Rank of A: Rank(A) =

1. # of nonzero rows in RREF
2. # of fixed variables
3. # of leading 1's in rref

X X X X X X X X XX X X X X

Matrix * Vector

A matrix A in $\mathbb{R}^{m \times n}$ * can only be multiplied by a vector in $\mathbb{R}^n$

(*) m rows & n columns

Suppose $A = \begin{bmatrix} | & | & & | \\ \vec{K}_1 & \vec{K}_2 & \cdots & \vec{K}_n \\ | & | & & | \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ $\quad v_i \in \mathbb{R}$, $\vec{K}_i \in \mathbb{R}^m$

Then $A\vec{v} = v_1\vec{K}_1 + v_2\vec{K}_2 + \dots + v_n\vec{K}_n$

And $A\vec{v} \in \mathbb{R}^m$ ($A\vec{v}$ gives a vector in $\mathbb{R}^m$)

X X X X X X X X X X X X X

Matrix * Matrix

For two matrices $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{q \times r}$

☆Note: if $q \neq n$, then you can't multiply AB.

Suppose $M = AB$

★ Note: $M = AB \not\Rightarrow M = BA$ (If $AB = BA$, A "commutes w/" B)

So how do we find M?

Assume $A = \begin{bmatrix} | & & | \\ \vec{k}_1 & \cdots & \vec{k}_n \\ | & & | \end{bmatrix}$ and $B = \begin{bmatrix} | & & | \\ \vec{c}_1 & \cdots & \vec{c}_r \\ | & & | \end{bmatrix}$ and

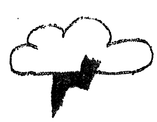
$M = \begin{bmatrix} | & & | \\ Q_1 & \cdots & Q_r \\ | & & | \end{bmatrix}$ $(M \in \mathbb{R}^{m \times r})$

Then $Q_i = A\vec{c}_i$

$$\iff M = \begin{bmatrix} | & | & & | \\ A\vec{c}_1 & A\vec{c}_2 & \cdots & A\vec{c}_r \\ | & | & & | \end{bmatrix}$$

General Solution of a Matrix A

1. Since ERO's preserve sol'n, find $RREF(A) = B$
2. The rows of B are equations, so solve for the dependent variables (columns w/ leading 1's)
3. Give the free variables funny names (s, t, ☺, , )
4. Express <u>all</u> (even free) variables in terms of funny names.
5. Put in the form:

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} * \\ \vdots \\ * \end{pmatrix} + $$  $$\begin{pmatrix} * \\ \vdots \\ * \end{pmatrix} + \cdots + $$  $$\begin{pmatrix} * \\ \vdots \\ * \end{pmatrix}$$

Section 2: Linear Transformations

- A LT T w/ domain $\mathbb{R}^n$ & codomain/range $\mathbb{R}^m$ is written "$T: \mathbb{R}^n \rightarrow \mathbb{R}^m$" and can be given by a matrix $A \in \mathbb{R}^{m \times n}$

- A transformation is linear $\Longleftrightarrow$
  1. $T(\vec{x} + \vec{v}) = T(\vec{x}) + T(\vec{v})$
  2. $T(c\vec{x}) = cT(\vec{x})$ for all $c \in \mathbb{R}$
  3. $T(\vec{0}) = \vec{0}$

How to find the matrix of a linear transformation T

If $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, then $T(\vec{x}) = T\left(\begin{bmatrix} x_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right) + T\left(\begin{bmatrix} 0 \\ x_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right) + \ldots + T\left(\begin{bmatrix} 0 \\ \vdots \\ x_n \end{bmatrix}\right)$

$= x_1 T(\vec{e}_1)^{*} + \ldots + x_n T(\vec{e}_n)$ (B/C T is linear)

☆ This looks a lot like Matrix * Vector

$= \underbrace{\begin{bmatrix} | & | & & | \\ T(e_1) & T(e_2) & \ldots & T(\vec{e}_n) \\ | & | & & | \end{bmatrix}}_{\text{This is } A!} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

So $A = \begin{bmatrix} | & | & & | \\ T(\vec{e}_1) & T(\vec{e}_2) & \ldots & T(\vec{e}_n) \\ | & | & & | \end{bmatrix}$

* e_i is the unit vector in the direction of the ith axis. $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix}$ $\vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$ etc.

How to find A^{-1}

$A \in \mathbb{R}^{m \times n}$ is invertible $\Leftrightarrow$

1. $m = n$
2. $\text{rank}(A) = m = n \Leftrightarrow \text{rref}(A) = I_n$ *

* $I_n \in \mathbb{R}^{n \times n}$ such that $I_n = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$

$A = I_n A = A I_n$

To find A^{-1}

- Augment A by I_n to get $[A \mid I_n]$
- Row reduce (everything done to the left should also be done to the right, as always)
- You should get $[I_n \mid ?]$
- $? = A^{-1}$

In $\mathbb{R}^{2 \times 2}$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

What does it mean?

$A^{-1} A = A A^{-1} = I_n$

So that's pretty cool...

Subspaces of $\mathbb{R}^n$

Def: $W \subset \mathbb{R}^n$ is a subspace $\iff$

① If $\vec{x} \in W$, then $c\vec{x} \in W$ for all $c \in \mathbb{R}$

② If $x, y \in W$ then $x+y \in W$

③ $\vec{0} \in W$

What is this "span" you speak of?

$$\text{span}\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n\} = \{c_1\vec{x}_1 + \dots + c_n\vec{x}_n \mid c_i \in \mathbb{R}\}$$

$= \{$set of ALL linear combos of $\vec{x}_1, \dots \vec{x}_n\}$

Kernel

$$\ker(A) = \{\vec{x} \in \mathbb{R}^m \mid A\vec{x} = T(\vec{x}) = 0\}$$

= Set of ALL $\vec{x}$'s in T's domain that return $\vec{0}$ as output

How to find it:

① Since ERO's preserve sol'n, find RREF(A)

② Find general sol'n of $[\text{RREF}(A) \mid \vec{0}]$

$= c_1\vec{x}_1 + \dots + c_k\vec{x}_k$ (See pg 3)

③ $\ker(A) = \text{span}\{\vec{x}_1, \vec{x}_2, \dots \vec{x}_k\}$

Image of $T(\vec{x}) = A\vec{x}$

$$\text{Image} = \{\vec{x} \in \mathbb{R}^n \mid \text{For some } \vec{v} \in \mathbb{R}^m,\ T(\vec{v}) = \vec{x}\}$$

Theory

$A(\vec{x})$ is always some linear combination of the column vectors of A, so strictly speaking

$$\text{Im}(A) = \text{span}\{\text{col vectors of } A\}$$

is correct, BUT we want a more economical way.

How-To

- Like normal, find rref(A)
- The leading 1s point out independent columns, so every column w/ a leading 1 is ind. in both A & rref(A)

So

$$\text{Im}(A) = \text{span}\{\text{columns of } A \text{ w/ a leading 1 in rref}(A)\}$$

Rank Nullity Thm / Fundamental Thm of Linear Algebra (FTOLA)

domain(A)

$$\dim(\text{Im}(A)) + \dim(\ker(A)) = \dim(\cancel{A})$$

$\dim(\text{Im}(A))$: # of Ind vectors in Im(A)

$\dim(\ker(A))$: # of ind. vectors in Ker(A)

$\dim(A)$: # of columns in A = # of variables