Math 235 Spring '09
Practice Mid-Term

1. Suppose $A \in \mathbb{R}^{n\times n}$ is an $n \times n$-matrix. Say in words what it means for $\vec{v}$ to be an element of $ker(A)$, and for $\vec{w}$ to be an element of $im(A)$. Show that $\vec{v} \in ker(A)$ implies that $\vec{v} \in ker(A^2)$. Does $\vec{w} \in im(A)$ imply that $\vec{w} \in im(A^2)$? Justify your answer.

2. Let $W \subset \mathbb{R}^4$ be the set of vectors $\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$ such that $x + y + z + w = 0$. Show that W is a subspace of $\mathbb{R}^4$. Find a basis for W. Express the vector $\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$ as a linear combination of your basis vectors.

3. a) Write the following system of equations as a matrix equation.

$$\begin{cases} x + y + z = 1 \\ 2x + 3z = 5 \\ 3x + y + 4z = 6 \end{cases}$$

   b) Solve the system using Gauss elimination.

4. a) Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{pmatrix}$, and $B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Do A and B commute, that is, is it true that $AB = BA$?

   b) Show that the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible if and only if $ad - bc \neq 0$.

5. a) What is the angle between the vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1/2 \\ \sqrt{3}/2 \end{pmatrix}$? (Hint: Dot product.)

b) Find the matrix of the rotation $T : \mathbb{R}^2 \to \mathbb{R}^2$ which takes $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} 1/2 \\ \sqrt{3}/2 \end{pmatrix}$.

6. Solve the linear system given in matrix form as

$$\begin{pmatrix} 0 & 2 & 3 \\ -1 & 1 & 1 \\ 0 & 3 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

by finding the inverse of the given matrix.

7. a) Check that the transformation defined by $T\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} -2y \\ x \end{smallmatrix}\right)$ is linear. Then find the matrix for this transformation.

   b) Find the inverse of the matrix from part (a).

8. Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 3 \\ 4 & 2 & 5 \end{pmatrix}$. Find equations in b_1, b_2, b_3 so that the system $A\vec{x} = \vec{b}$, where $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, has a solution. Find a basis of the image of A.

9. True of false? Explain.

   a) If the reduced echelon form of a matrix A has a row of zeros, it must have a non-zero kernel.

   b) If the column vectors of a matrix B form a basis of its image, it must be that $ker(B) = \{\vec{0}\}$.

   c) If A is an invertible matrix, and B and C any two matrices such that $AB = AC$, then it must in fact be that $B = C$.