Inner Product Spaces, Orthogonal Projections, and Orthonormal Bases

Def: An inner product on a vector space V is a map \( \langle \cdot, \cdot \rangle : V \times V \to \mathbb{R} \) such that

(a) \( \langle f, g \rangle = \langle g, f \rangle \) (symmetry)

(b) \( \langle f + g, h \rangle = \langle f, h \rangle + \langle g, h \rangle \)

(c) \( \langle cf, g \rangle = c \langle f, g \rangle \) \( \text{LT in 1\textsuperscript{st} coordinate.} \)

(d) \( \langle f, f \rangle \geq 0 \) for all \( f \neq 0 \) in V (positive definite)

Note: By (c), we can see that \( \langle f, f \rangle = 0 \) if and only if \( f = 0 \).

Note: Applying the symmetry condition (a) to conditions (b) and (c), we see that

\[ \langle f, g + h \rangle = \langle f, g \rangle + \langle f, h \rangle \text{ and } \langle f, cg \rangle = c \langle f, g \rangle. \]

So inner products are linear transformations in both coordinates.

Example: The dot product on \( \mathbb{R}^n \) is the most common example of an inner product. Show that the dot product is an inner product.

(a) \( \langle \mathbf{v}, \mathbf{w} \rangle = v_1w_1 + v_2w_2 + \cdots + v_nw_n \)

(b) \( \langle \mathbf{v} + \mathbf{w}, \mathbf{u} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle + \langle \mathbf{w}, \mathbf{u} \rangle \)

(c) \( \langle c\mathbf{v}, \mathbf{w} \rangle = c\langle \mathbf{v}, \mathbf{w} \rangle \)

(d) \( \langle \mathbf{v}, \mathbf{v} \rangle = \mathbf{v}_1^2 + \mathbf{v}_2^2 + \cdots + \mathbf{v}_n^2 = 0 \iff \mathbf{v}_1 = \cdots = \mathbf{v}_n = 0 \). So \( \langle \mathbf{v}, \mathbf{v} \rangle > 0 \) for all \( \mathbf{v} \neq 0 \).

Example: Let \( V = \mathbb{R}^{m \times n} \), the space of all \( m \times n \) matrices with entries in \( \mathbb{R} \). Define a pairing on \( V \) by \( \langle A, B \rangle = \text{tr}(A^T B) \) for all matrices \( A, B \) in \( V \). Show that \( \langle \cdot, \cdot \rangle \) is an inner product.

(a) \( \langle A^T B, C^T D \rangle = \langle C D^T A^T, B \rangle \)

(b) \( \langle A+C, B \rangle = \langle A, B \rangle + \langle C, B \rangle \)

(c) \( \langle A \rangle = \langle A^T B, B \rangle = \text{tr}(A^T B) \)

(d) \( \text{If } A \neq 0 \text{, show } A = [v_1 \ldots v_m]. \text{ Then } \langle A, A \rangle = \text{tr} \left( \begin{bmatrix} -v_1 & v_2 & \cdots & v_m \end{bmatrix} \begin{bmatrix} v_1 & \cdots & v_m \end{bmatrix} \right) = \sum_{i=1}^{m} \sum_{j=1}^{m} v_i v_j = \mathbf{v}_1^2 + \cdots + \mathbf{v}_m^2 \neq 0. \)

Def: The norm or magnitude of an element \( f \) in an inner product space \( V \) is:

\[ \| f \| = \sqrt{\langle f, f \rangle} \]

Def: We say that two elements \( f, g \) in an inner product space \( V \) are orthogonal or perpendicular if: \( \langle f, g \rangle = 0 \).

Def: The distance between two elements \( f, g \) in an inner product space \( V \) is:

\[ \| f - g \| = \sqrt{\langle f-g, f-g \rangle} \]

Example: Let \( V = C[0,1] \), and consider the inner product \( \langle f, g \rangle = \int_0^1 f(t)g(t) \, dt \). Let \( m(t) = 3t^2 \) and \( n(t) = -t \).

Find \( \| m \| \) and find the distance between \( m \) and \( n \).

(a) \( \| m \| = \sqrt{\langle m, m \rangle} = \sqrt{\int_0^1 m(t)^2 \, dt} = \sqrt{\int_0^1 (3t^2)^2 \, dt} = \sqrt{\frac{9}{4} t^4} \bigg|_0^1 = \sqrt{\frac{9}{4}} \)

(b) \( \| m - n \| = \| 3t^2 + t \| = \sqrt{\int_0^1 (3t^2 + t)^2 \, dt} = \sqrt{\frac{9}{4} t^4 + \frac{6}{3} t^3 + \frac{1}{3} t^2} \bigg|_0^1 = \sqrt{\frac{27}{10}} \)
Note: The standard inner product on \( \mathbb{R}^n \) is the dot product. So in \( \mathbb{R}^n \), we have the following facts.

- Two vectors \( \vec{u}, \vec{v} \in \mathbb{R}^n \) are orthogonal/perpendicular if \( \vec{u} \cdot \vec{v} = 0 \).
- The length of a vector \( \vec{v} \in \mathbb{R}^n \) is \( ||\vec{v}|| = \sqrt{\vec{v} \cdot \vec{v}} \).
- A vector \( \vec{u} \in \mathbb{R}^n \) is a unit vector if the length of \( \vec{u} \) is 1, i.e., if \( ||\vec{u}|| = 1 \).
- If \( \vec{u} \in \mathbb{R}^n \), then the vector \( \vec{u} = \frac{\vec{v}}{||\vec{v}||} \) is a unit vector in the same direction as \( \vec{v} \).
- A vector \( \vec{u} \in \mathbb{R}^n \) is orthogonal to a subspace \( V \) in \( \mathbb{R}^n \) if \( \vec{u} \) is orthogonal to every vector in \( V \), i.e., if \( \vec{u} \cdot \vec{v} = 0 \) for all \( \vec{v} \in V \).

Def: We say that a collection \( g_1, \ldots, g_m \) of elements in an inner product space \( V \) are orthonormal if they are unit vectors and if each one is orthogonal to the rest. In other words, the collection is orthonormal if

\[
\langle g_i, g_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}
\]

Ex: The standard basis vectors \( \vec{e}_1, \ldots, \vec{e}_n \in \mathbb{R}^n \) are orthonormal. (So they form an orthonormal basis of \( \mathbb{R}^n \)). Also, any subset of this collection is orthonormal.

Ex: Let \( \theta \) be any angle. Show that the vectors \( \begin{bmatrix} \sin \theta \\ \cos \theta \end{bmatrix} \) and \( \begin{bmatrix} -\cos \theta \\ \sin \theta \end{bmatrix} \) are orthonormal in \( \mathbb{R}^2 \).

Unit vectors:
\[
\vec{u} \cdot \vec{v} = \sin^2 \theta + \cos^2 \theta = 1
\]
\[
\vec{w} \cdot \vec{w} = \cos^2 \theta + \sin^2 \theta = 1
\]

Orthogonal:
\[
\vec{u} \cdot \vec{v} = -\sin \theta \cos \theta + \cos \theta \sin \theta = 0.
\]

Ex: Show that the vectors
\[
\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}
\]
are orthonormal in \( \mathbb{R}^2 \).

Unit:
\[
\vec{u} \cdot \vec{u} = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1
\]
\[
\vec{w} \cdot \vec{w} = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1
\]

Orthogonal:
\[
\vec{u} \cdot \vec{w} = \frac{1}{4} - \frac{1}{4} + \frac{1}{4} - \frac{1}{4} = 0.
\]

Thm: Orthogonal vectors are linearly independent. So any set of \( n \) orthogonal vectors in a vector space \( V \) of dimension \( n \) is a basis for \( V \).

Recall: (Ch. 2) Let \( L \) be any line in \( \mathbb{R}^2 \). Any vector \( \vec{z} \in \mathbb{R}^2 \) can be written uniquely as \( \vec{z} = \vec{z}_p + \vec{z}_\perp \), where \( \vec{z}_p \) is parallel to \( L \) and \( \vec{z}_\perp \) is orthogonal to \( L \). Here \( \vec{z}_p \) is also denoted \( \text{proj}_L(\vec{z}) \).

Thm: Suppose \( g_1, \ldots, g_m \) is an orthonormal basis of a subspace \( W \) of an inner product space \( V \). Then the projection of an element \( f \) in \( V \) onto the subspace \( W \) is:

\[
\text{proj}_W f = \langle g_1, f \rangle g_1 + \cdots + \langle g_m, f \rangle g_m.
\]
Fact: Let $W$ be a subspace of $\mathbb{R}^n$. Let $\{\vec{u}_1, \ldots, \vec{u}_m\}$ be an orthonormal basis of $V$. Then for any $\vec{x} \in \mathbb{R}^n$,
\[
\text{proj}_W(\vec{x}) = \vec{x}_W = (\vec{x} \cdot \vec{u}_1) \vec{u}_1 + \cdots + (\vec{x} \cdot \vec{u}_m) \vec{u}_m = \sum_{i=1}^{m} (\vec{x} \cdot \vec{u}_i) \vec{u}_i.
\]

Obs: Let $V$ be the $(xy)$-plane in $\mathbb{R}^3$. Then $\{\vec{e}_1, \vec{e}_2\}$ is an orthonormal basis for $V$. Let $L_1 = \text{span}\{\vec{e}_1\}$ be the $x$-axis and let $L_2 = \text{span}\{\vec{e}_2\}$ be the $y$-axis. Then for any $\vec{x} \in \mathbb{R}^3$, $\text{proj}_V(\vec{x}) = \text{proj}_{L_1}(\vec{x}) + \text{proj}_{L_2}(\vec{x})$.

This is true in general: Let $V$ be a subspace of $\mathbb{R}^n$. Let $\{\vec{u}_1, \ldots, \vec{u}_m\}$ be an orthonormal basis of $V$. Let $L_i = \text{span}(\vec{u}_i)$ for each $i$. Then
\[
\text{proj}_V(\vec{x}) = \sum_{i=1}^{m} (\vec{u}_i \cdot \vec{x}) \vec{u}_i = \sum_{i=1}^{m} \text{proj}_{L_i}(\vec{x}).
\]

Ex: Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$ and let $V = \text{Im}(A)$. Find $\text{proj}_V(\vec{x})$ for $\vec{x} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$.

1. Find an orthonormal basis for $V = \text{Im}(A)$:
$\vec{u}_1 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

2. $\text{proj}_V(\vec{x}) = (\vec{x} \cdot \vec{u}_1) \vec{u}_1 + (\vec{x} \cdot \vec{u}_2) \vec{u}_2 = \left( \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \right) \vec{u}_1 + \left( \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \right) \vec{u}_2 = \vec{u}_1 + \vec{u}_2$.

Thm: Let $\vec{u}_1, \ldots, \vec{u}_n$ be an orthonormal basis for $\mathbb{R}^n$. Then for any $\vec{x} \in \mathbb{R}^n$, $\vec{x} = (\vec{u}_1 \cdot \vec{x}) \vec{u}_1 + \cdots + (\vec{u}_n \cdot \vec{x}) \vec{u}_n (\equiv \text{proj}_{\text{span}(\vec{u}_1})(\vec{x})))$.

Note: This means that if $\vec{u}_1, \ldots, \vec{u}_n$ is an orthonormal basis for $\mathbb{R}^n$, then $\vec{x} = \sum_{i=1}^{n} \text{proj}_{\text{span}(\vec{u}_i)}(\vec{x})$ for any $\vec{x} \in \mathbb{R}^n$.

Note: The theorem means that if $B$ is an orthonormal basis for $\mathbb{R}^n$, then we now have a formula for finding the $B$-coordinates of any $\vec{x} \in \mathbb{R}^n$.

Ex: Consider the orthonormal basis $B = \left\{ \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}, \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \end{bmatrix} \right\}$ of $\mathbb{R}^3$. Find the $B$-coordinates of $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$.

Recall: $[\vec{x}]_B = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ where $\vec{x} = a \vec{u}_1 + b \vec{u}_2 + c \vec{u}_3 + d \vec{u}_4$.

Old method: solve these four linear equations.

New method: define $[\vec{x}]_S = \begin{bmatrix} 5 \\ 0 \\ -1 \\ -2 \end{bmatrix}$.
Recall: Let \( V \) be a subspace of \( \mathbb{R}^n \). The orthogonal complement of \( V \) is the set \( V^\perp \) of all vectors \( \bar{x} \in \mathbb{R}^n \) which are orthogonal to \( V \). (Note that \( V^\perp \) is the kernel of the orthogonal projection \( \text{proj}_V \) onto \( V \)).

Properties: Let \( V \) be a subspace of \( \mathbb{R}^n \).

1. Then \( V^\perp \) is a subspace of \( \mathbb{R}^n \).
2. \( V \cap V^\perp = \{0\} \).
3. \( \dim(V) + \dim(V^\perp) = n \).
4. \( (V^\perp)^\perp = V \).

Pythagorean Theorem: Let \( \bar{x}, \bar{y} \in \mathbb{R}^n \). Then \( \bar{x} \perp \bar{y} \) if and only if \( ||\bar{x} + \bar{y}||^2 = ||\bar{x}||^2 + ||\bar{y}||^2 \).

Proof:

\[
||\bar{x} + \bar{y}||^2 = (\bar{x} + \bar{y}) \cdot (\bar{x} + \bar{y}) = \bar{x} \cdot \bar{x} + \bar{x} \cdot \bar{y} + \bar{y} \cdot \bar{x} + \bar{y} \cdot \bar{y}
\]
\[
= ||\bar{x}||^2 + 2\bar{x} \cdot \bar{y} + ||\bar{y}||^2
\]
\[
= ||\bar{x}||^2 + 0 + ||\bar{y}||^2
\]

\( \implies \)

\( \bar{x} \cdot \bar{y} = 0 \) (\( \bar{x} \perp \bar{y} \)).
Section 5.2: Gram-Schmidt Process and QR Factorization

**Method:** (The Gram-Schmidt Process) Let \( \vec{v}_1, \ldots, \vec{v}_n \) be a basis for a subspace \( V \) in \( \mathbb{R}^n \). Let

\[
\begin{align*}
\vec{w}_1 &= \vec{v}_1 \\
\vec{w}_2 &= \vec{v}_2 - \frac{\vec{v}_2 \cdot \vec{w}_1}{\vec{w}_1 \cdot \vec{w}_1} \vec{w}_1 \\
\vec{w}_3 &= \vec{v}_3 - \frac{\vec{v}_3 \cdot \vec{w}_1}{\vec{w}_1 \cdot \vec{w}_1} \vec{w}_1 - \frac{\vec{v}_3 \cdot \vec{w}_2}{\vec{w}_2 \cdot \vec{w}_2} \vec{w}_2 \\
&\vdots
\end{align*}
\]

Then \( \vec{w}_1, \ldots, \vec{w}_n \) is an orthogonal basis for \( V \).

Let \( \vec{u}_1 = \frac{\vec{w}_1}{||\vec{w}_1||}, \vec{u}_2 = \frac{\vec{w}_2}{||\vec{w}_2||}, \ldots \) Then \( \vec{u}_1, \ldots, \vec{u}_n \) is an orthonormal basis for \( V \).

**Ex:** Let \( B = \left\{ \vec{v}_1 = \begin{bmatrix} -3 \\ 4 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \). Then \( B \) is a basis for \( \mathbb{R}^2 \). Use the Gram-Schmidt process to find an orthonormal basis \( \mathcal{U} = \{ \vec{u}_1, \vec{u}_2 \} \) of \( \mathbb{R}^2 \).

\[
\begin{align*}
\vec{w}_1 &= \vec{v}_1 = \begin{bmatrix} -3 \\ 4 \end{bmatrix} \\
\vec{w}_2 &= \vec{v}_2 - \frac{\vec{v}_2 \cdot \vec{w}_1}{\vec{w}_1 \cdot \vec{w}_1} \vec{w}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{13}{17} \begin{bmatrix} -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 13/17 \\ 6/17 \end{bmatrix} \\
\vec{u}_1 &= \frac{\vec{w}_1}{\|\vec{w}_1\|} = \frac{1}{\sqrt{17}} \begin{bmatrix} -3 \\ 4 \end{bmatrix} \\
\vec{u}_2 &= \frac{\vec{w}_2}{\|\vec{w}_2\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 13/17 \\ 6/17 \end{bmatrix} = \begin{bmatrix} 13/17 \\ 6/17 \end{bmatrix}
\end{align*}
\]

Then \( \mathcal{U} = \{ \vec{u}_1, \vec{u}_2 \} \) is an orthonormal basis for \( \mathbb{R}^2 \) (can check).

**Ex:** Let \( V = \text{Im}(M) \) where \( M = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \).

1. Find an orthonormal basis for \( V \).

2. Find the change of basis matrix \( R \) from the basis \( B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\} \) to the basis \( \mathcal{U} \) found in 1.

\[
\begin{align*}
1. \quad \vec{w}_1 &= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\
\vec{w}_2 &= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \\ 0 \end{bmatrix}
\end{align*}
\]

(So \( \vec{v}_1, \vec{v}_2 \) were already orthonormal).

\[
\begin{align*}
\vec{u}_1 &= \frac{\vec{w}_1}{\|\vec{w}_1\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\
\vec{u}_2 &= \frac{\vec{w}_2}{\|\vec{w}_2\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/2 \sqrt{2} \\ 0 \end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\mathcal{U} &= \left\{ \begin{bmatrix} 1/\sqrt{2} \\ 1/2 \sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 1/2 \sqrt{2} \\ 0 \end{bmatrix} \right\} \\
R &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}
\end{align*}
\]
Thm: \((QR\)-factorization\) Let \(M\) be an \(n \times m\) matrix with linearly independent columns \(\vec{v}_1, \ldots, \vec{v}_m\). Then there exists an \(n \times m\) matrix \(Q\) whose columns \(\vec{u}_1, \ldots, \vec{u}_m\) are orthonormal, and an upper triangular matrix \(R\) with positive diagonal entries, such that

\[ M = QR \]

Furthermore, \(r_{jj} = \|\vec{u}_j\|, r_{ij} = \|\vec{v}_j\|\) for \(2 \leq j \leq m\), and \(r_{ij} = \vec{u}_i \cdot \vec{v}_j\) for \(i < j\).

Ex: Verify the \(QR\)-factorization theorem for the last example.

Let \(Q = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ \vec{v}_4 & \vec{v}_5 & \vec{v}_6 \end{bmatrix}, R = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 7 \end{bmatrix}\).

Then \(QR = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 7 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 6 \\ 0 & 0 \end{bmatrix} = M\).

Ex: Find the \(QR\)-factorization of \(M = \begin{bmatrix} 2 & 2 \\ 1 & 1 \\ -2 & 1 \end{bmatrix}\).

\(\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}\).

A) Gram-Schmidt to find columns \(\vec{u}_i\) of \(Q\):

1. \(\vec{\omega}_1 = \vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}\), \(\vec{\omega}_2 = \vec{v}_2 - \frac{\vec{v}_2 \cdot \vec{\omega}_1}{\vec{\omega}_1 \cdot \vec{\omega}_1} \vec{\omega}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} - \frac{9}{5} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2/5 \\ 4/5 \\ 9/5 \end{bmatrix}\).

2. \(\vec{u}_1 = \frac{\vec{\omega}_1}{\|\vec{\omega}_1\|} = \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix}\), \(\vec{u}_2 = \frac{\vec{\omega}_2}{\|\vec{\omega}_2\|} = \begin{bmatrix} 2/5 \\ 4/5 \\ 9/5 \end{bmatrix} = \begin{bmatrix} 2/5 \\ 4/5 \\ 9/5 \end{bmatrix}\).

\(Q = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 \end{bmatrix} = \begin{bmatrix} 2/3 & 2/5 \\ 1/3 & 4/5 \\ -2/3 & 9/5 \end{bmatrix}\).

B) \(R = \begin{bmatrix} [\vec{v}_1]_{\vec{u}_1} & [\vec{v}_2]_{\vec{u}_1} \\ [\vec{v}_1]_{\vec{u}_2} & [\vec{v}_2]_{\vec{u}_2} \end{bmatrix}\):

\([\vec{v}_1]_{\vec{u}_1} = (\vec{u}_1, \vec{v}_1) \vec{u}_1 = 3 \vec{u}_1 + 0 \vec{u}_2 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}\).

\([\vec{v}_2]_{\vec{u}_1} = (\vec{u}_1, \vec{v}_2) \vec{u}_1 = 3 \vec{u}_1 + \sqrt{5} \vec{u}_2 = \begin{bmatrix} 3 \\ \sqrt{5} \end{bmatrix}\).

\(R = \begin{bmatrix} 3 & 3 \\ 0 & \sqrt{5} \end{bmatrix}\).

C) Check! \(QR = MV\).
Section 5.3: Orthogonal Transformations and Orthogonal Matrices

Def: A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is called an orthogonal linear transformation if $T$ preserves the lengths of vectors, i.e. if

$$
\| T(\vec{x}) \| = \| \vec{x} \| \quad \text{for all} \quad \vec{x} \in \mathbb{R}^n
$$

Ex: The rotation by $\theta$ matrix $T(\vec{x}) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \vec{x}$ of $\mathbb{R}^2$ is an orthogonal transformation.

Ex: Let $V$ be a subspace of $\mathbb{R}^n$. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the transformation which reflects each vector across the subspace $V$. Then for all $\vec{x} \in \mathbb{R}^n$, $T(\vec{x}) = \text{ref}_V(\vec{x}) = \vec{x}_0 - \vec{x}_0$. Show that $T$ is orthogonal.

$$
\| T(\vec{x}) \| = \| \vec{x}_0 - \vec{x}_0 \| = \| \vec{x}_0 \| = \| \vec{x}_0 \| = \| \vec{x}_0 + \vec{x}_0 \| \quad \text{by Pyth. Thm}.
$$

Thm: 1. A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal if and only if the vectors $T(\vec{e}_1), \ldots, T(\vec{e}_n)$ form an orthonormal basis for $\mathbb{R}^n$.

2. An $n \times n$ matrix $A$ is orthogonal if and only if its columns form an orthonormal basis of $\mathbb{R}^n$.

Ex: Is the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ orthogonal?

No: the columns are orthogonal, but they aren't unit vectors.

Ex: Is the matrix $A = \frac{1}{2} \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$ orthogonal?

Yes: the columns are orthogonal and they are unit vectors, so the columns are orthonormal $\Rightarrow A$ is orthogonal.

Thm: Suppose $A$ and $B$ are orthogonal matrices. Then $A^{-1}$ and $AB$ are orthogonal. In other words, inverses and products of orthogonal matrices are orthogonal.

Def: An $n \times n$ matrix $A$ is symmetric if $A^T = A$, and skew-symmetric if $A^T = -A$.

Ex: The matrix $A$ below is symmetric and the matrix $B$ below is skew-symmetric.

$$
A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 2 & 3 \\ 3 & 2 & 2 & 1 \\ 4 & 3 & 1 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{bmatrix}
$$
Ex: Find all $2 \times 2$ skew-symmetric matrices.

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is skew-symmetric, then $A^T = -A$, i.e., $\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} -a & b \\ -c & -d \end{bmatrix}$

$a = -a \Rightarrow a = 0$
$d = -d \Rightarrow d = 0$

$\Rightarrow A = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$

$10 = -c$

Thm: An $n \times n$ matrix $A$ is orthogonal if and only if $A^T A = I_n$, i.e., if and only if $A^{-1} = A^T$.

Summary: Let $A$ be an $n \times n$ matrix. Then the following are equivalent:

1. $A$ is an orthogonal matrix.
2. The transformation $L(\mathbf{x}) = A\mathbf{x}$ preserves length, $||A\mathbf{x}|| = ||\mathbf{x}||$ for every $\mathbf{x} \in \mathbb{R}^n$. (def.)
3. The columns of $A$ form an orthonormal basis of $\mathbb{R}^n$. (Thm 1)
4. $A^T A = I_n$. (Def 2)
5. $A^{-1} = A^T$. (Def 2)

Properties: of $A^T$ and $A^{-1}$: Let $A$ and $B$ be $n \times n$ matrices. If $A$ is symmetric, $B$ is skew-symmetric, $A + B$ commutes, $\text{then} B^T A = (-B)A = -(AB)$

Thm: Let $V$ be a subspace of $\mathbb{R}^n$ with orthonormal basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$. The linear transformation $T(\mathbf{x}) = \text{proj}_V(\mathbf{x})$ has standard matrix $QQ^T$ where $Q = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}$.

Ex: Find the matrix of orthogonal projection onto the subspace of $\mathbb{R}^4$ spanned by the vectors

$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, \hspace{1cm} $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$

$B = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}$ is an orthonormal basis of this subspace.

So $\text{proj}_V$ has matrix $QQ^T = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \\ \mathbf{v}_2 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

i.e., $\text{proj}_V(\mathbf{x}) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^4$. 