CANONICAL SPLITTINGS FOR 2×2 MATRICES (AFTER ARUNAS RUDVALIS)

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Suppose $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{2\times 2}$ is a real 2×2 matrix with *trace* $\tau = a + d$ and *determinant* $\delta = ad - bc$. There are three cases to consider, depending on the sign of the discriminant $(a + d)^2 - 4(ad - bc) = \tau^2 - 4\delta$ of the *characteristic polynomial* of A whose roots $\lambda = \frac{\tau+\sqrt{\tau^2-4\delta}}{2}$ and $\mu = \frac{\tau-\sqrt{\tau^2-4\delta}}{2}$ are the *eigenvalues* of A:

- These are real and distinct $\lambda > \mu$ if and only if $\tau^2 - 4\delta > 0$, and in this case there is an invertible 2×2 matrix $E \in \mathsf{GL}(2, \mathbb{R})$ such that $A = E\Delta E^{-1}$ where $\Delta = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ with $\lambda + \mu = a + b = \tau$ and $\lambda\mu = (ad - bc) = \delta$. The columns of the matrix E are *eigenvectors* for A, that is, nonzero elements of $\ker(A - \lambda I)$ and $\ker(A - \mu I)$, corresponding to λ and μ, respectively.
- These are real and equal $\lambda = \mu$ if and only if $\tau^2 - 4\delta = 0$, and in this case we can write $A = \lambda I + N$ where $\lambda = (a + d)/2 = \tau/2$ and $N = A - \lambda I$ satisfies $N^2 = 0$.
- These are a complex conjugate pair $\lambda = \alpha + i\beta \neq \mu = \bar{\lambda} = \alpha - i\beta$ (with $\beta \neq 0$) if and only if $\tau^2 - 4\delta < 0$, and in this case we can write $A = \alpha I + \beta J$ where $\alpha = (a + d)/2 = \tau/2$, $\beta = \sqrt{4\delta - \tau^2}/2$, and $J = (A - \alpha I)/\beta$ satisfies $J^2 = -I$.

Each of these allows the rapid calculation of powers (or other functions representable by power series, such as the exponential, which is important for solving differential equations) of A. For example, in the middle case we have $A^p = (\lambda I + N)^p = \lambda^p I + p\lambda^{p-1}N$ since $N^2 = 0$, and in the last case $A^p = (\alpha I + \beta J)^p = *I + *J$ where the expressions $*$ are computed by the binomial formula using $J^2 = -I, J^3 = -J, J^4 = I \ldots$.

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Date: December 10, 2013, revised September 4, 2019.