DERIVATIVES, CRITICAL POINTS AND CONVEXITY

The following problems concern the functions

$$h(x,y) := xy,\ e(x,y) := (\sin x)(\sin y),\ f(x,y) := \sin(xy) \text{ and } p(x,y) := x^5 - y^4$$

on $\mathbb{R}^2$ (or on the specified subdomains). We have explored h and p in class, whose graphs resemble a saddle and the weird pen-holder we passed around in class; the graph of e should remind you of the egg-crate we also passed around, and you are welcome to draw or describe the graph of f for extra credit!

Problem 0.1. *Suppose you're hiking along the graph of the function h above the path $(x(t), y(t)) := (t^2, t^3)$ in $\mathbf{R}^2$. At what rate is your height $z(t) = h(x(t), y(t))$ changing when $t = 2$? (You can do this various ways, but make sure one of the ways uses the chain rule and directional derivatives!) For what value(s) of t does the height rate of change vanish?*

Problem 0.2. *For the functions e and f above, compute their various first and second partial derivatives (as functions of (x, y)), as well as their gradient vector fields and Hessian matrix fields (whose entries are also functions of (x, y) – we did this in class for the functions h and p). Find all the critical points for both e and f, and use their Hessians there to analyze whether these critical points are (local) minima, maxima, saddles or otherwise.*

Problem 0.3. *We saw (in class) that the only critical point of the function p is at $(0, 0)$ (it's degenerate – its Hessian is the zero matrix there), but we can still use the sign of its Hessian to find where the graph of p is convex-up, convex-down (concave) or otherwise – please do that!*

Problem 0.4. *Find all the global maximum and global minimum* points *and* values *of the function h above restricted to the* unit square $Q^2 := [0, 1] \times [0, 1]$. *Do the same for the* unit disk $D^2 = \{(x, y) \mid x^2 + y^2 \leq 1\}$. *(This problem can be done in various ways, but the method of Lagrange multipliers should be used and compared with other methods you try.)*

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