INTEGRALS, VOLUMES AND CENTERS OF MASS

In the problems below, we continue using the familiar functions

$$h(x,y) := xy, \ \ e(x,y) := (\sin x)(\sin y), \ \text{ and } \ p(x,y) := x^5 - y^4,$$

and the familiar subdomains of the (x,y)-plane $\mathbf{R}^2$:

- the *unit square* $Q^2 := [0,1] \times [0,1] = \{(x,y) \mid 0 \le x, \ y \le 1\}$,
- the *unit disk* $D^2 := \{(x,y) \mid x^2 + y^2 \le 1\}$, and
- the *first quadrant* of the unit disk $Q^2 \cap D^2 = \{(x,y) \mid 0 \le x, \ y, \ x^2 + y^2 \le 1\}$.

We will explore the 3-dimensional regions R between their graphs in $\mathbf{R}^3$ and various subdomains P of $\mathbf{R}^2$ which we view as the plane $\{z = 0\} \subset \mathbf{R}^3$.

For *each* problem:

- Express the volume of R as an integral over P, and evaluate it as an iterated integral. [Hint: for some problems we may need to figure out P, or make substitutions, or use polar coordinates...!]
- Find the center of mass $(\bar{x}, \bar{y}, \bar{z})$ for the region R, where $\bar{f}$ means the *average* of the function f over the region R. [Hint: symmetry helps check $\bar{x}$ or $\bar{y}$ without computation of integrals; $\bar{z}$ is trickier to compute: reduce it to (a multiple of) the integral over P of the *square* of the function whose graph defines R!]

Problem 0.1. *Let $R \subset \mathbf{R}^3$ be the region between $P = Q^2 = [0,1] \times [0,1] \subset \mathbf{R}^2$ and* $\operatorname{graph}(h)$.

[Hint: both h and $P = Q^2$ are symmetric with respect to the line $\{x = y\}$!]

Problem 0.2. *Let $R \subset \mathbf{R}^3$ be the region between $P = Q^2 \cap D^2$ and* $\operatorname{graph}(h)$.

[Hint: consider polar coordinates as well as symmetry with respect to $\{x = y\}$!]

Problem 0.3. *Let $R \subset \mathbf{R}^3$ be the region between $P = \pi Q^2 = [0,\pi] \times [0,\pi] \subset \mathbf{R}^2$ (the unit square rescaled by π) and* $\operatorname{graph}(e)$.

[Hint: symmetry with respect to $\{x = y\}$ as well as $\{x = \frac{\pi}{2}\}$ and $\{y = \frac{\pi}{2}\}$; the integral $\int \sin^2 t \, dt$ over a half-period is easy to compute!]

Problem 0.4. *Let $R \subset \mathbf{R}^3$ be the region between* $\operatorname{graph}(p)$ *and $P = \{(x,y) | \ p(x,y) \ge 0\} \cap Q^2$, the subdomain of the unit square where p is non-negative.*

[Hint: first find the curved part of the boundary of P where $p(x,y) = 0$; this gives limits of integration depending on the order one does the iterated integral; one order is much easier than the other, but Fubini's Theorem certifies they are equal (you can try both orders and verify this if you like)!]

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