GRADIENT, DIVERGENCE AND LAPLACIAN IN n-SPACE

Recall that **grad** $= \nabla$ is the *gradient* operator taking smooth functions to smooth vectorfields, that **div** $= \nabla\cdot$ is the *divergence* operator taking smooth vectorfields to smooth functions, and that their composition **div**$\circ$**grad** $= \nabla \cdot \nabla =: \Delta$ is the *Laplace* operator (or *Laplacian*) taking smooth functions to smooth functions; in particular, if f is a function on a domain $R \subset \mathbf{R}^n$ then

$$\Delta f = f_{11} + f_{22} \cdots + f_{nn}$$

where f_{ii} denotes the second partial derivative of f with respect to the i-th coordinate x_i on $\mathbf{R}^n$.

Problem 0.1. *There are various* product rules *(in the spirit of Leibniz) for the above operators, for example $\nabla(fg) = (\nabla f)g + f(\nabla g)$ for any smooth functions f, g on $R \subset \mathbf{R}^n$. Prove this formula and work out analogous formulas for $\nabla \cdot (f\mathbf{V})$ and for $\Delta(fg)$ where $\mathbf{V}$ is any smooth vectorfield on R. (For* extra credit*, try to figure out formulas for $\nabla(\mathbf{V} \cdot \mathbf{W})$ and for $\Delta(\mathbf{V} \cdot \mathbf{W})$ where $\mathbf{W}$ is another smooth vectorfield on R.)*

Problem 0.2. *Let $r = r(x_1, \cdots, x_n) = r(\mathbf{x}) = ||\mathbf{x}|| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$ be the radial distance function on $\mathbf{R}^n$. By applying the chain rule to $\nabla(r^2) = 2\mathbf{x}$, we computed in class that $\nabla r = \mathbf{x}/r =: \mathbf{u}$ is the outward unit radial vectorfield on the domain $R = \mathbf{R}^n - \{\mathbf{0}\}$ (sketch a picture of $\mathbf{u}$ to remind yourself of the issue at $\mathbf{0}$). For an arbitrary radial function $f = f(r) = f(r(\mathbf{x}))$, carry out a similar computation to first express ∇f, and then $\Delta f = \nabla \cdot (\nabla f)$, in terms of $\mathbf{u}$, $f_r = f'(r)$, and $f_{rr} = f''(r)$.*

[Hint: You should find $\Delta f = f_{rr} + \frac{n-1}{r} f_r$. The "correction" term $\frac{n-1}{r} f_r$ has a geometric meaning. When n=2 it's the "curvature" of the circle of radius r around the origin.]

Problem 0.3. *Use your result from the previous problem to find the* harmonic *radial functions on the domain $R = \mathbf{R}^n - \{\mathbf{0}\}$, that is, those functions $f = f(r) = f(r(\mathbf{x}))$ so that $\Delta f = 0$.*

[Hint: Try using a function of the form $f(r) = r^p$. Your formula for Δf should involve a quadratic expression in p; roots depend on the dimension n and give the powers p making $f(r) = r^p$ harmonic; one of the roots is always zero, which makes sense since $f =$ constant is always harmonic; this works for all $n \neq 2$, but something special happens for $n = 2$, and the radial function $\log r$ appears....]

Problem 0.4. *We saw in class that the function $e(x, y) := (\sin x)(\sin y)$ is an* eigenfunction *of the Laplacian on the rescaled square $R = \pi Q^2 := [0, \pi] \times [0, \pi]$ — it satisfies the* boundary value problem

$$\Delta e + \lambda e = 0 \ \textit{on}\ R, \quad e = 0 \ \textit{on}\ \partial R$$

— with eigenvalue *$\lambda = 2$. Verify that for any pair of integers $k, l > 0$ the function $e^{k,l}(x, y) := (\sin kx)(\sin ly)$ is also an eigenfunction of the Laplacian (with eigenvalue $\lambda^{k,l} = k^2 + l^2$), and investigate the* spectrum *(the increasing list of eigenvalues with their multiplicities) of the Laplacian in this case. For* fun *or* extra credit*, also try this in case of the rescaled n-cube $R = \pi Q^n$ or n-rectangle $R = [0, a_1] \times \cdots \times [0, a_n]$.*

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