CURLING VECTORFIELDS IN 3-SPACE

Recall that the **curl** $= \nabla\times$ operator taking smooth vectorfields to smooth vectorfields on a domain $R \subset \mathbf{R}^3$ is defined by

$$\mathbf{curl}(\mathbf{V}) = \nabla \times \mathbf{V} = (\partial_1, \partial_2, \partial_3) \times (v_1, v_2, v_3) := \mathbf{e}_1(\partial_2 v_3 - \partial_3 v_2) + \mathbf{e}_2(\partial_3 v_1 - \partial_1 v_3) + \mathbf{e}_3(\partial_1 v_2 - \partial_2 v_1)$$

for any vectorfield $\mathbf{V} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3$ on R.

As the notation suggests, it can be useful to think of **curl** as the cross product of the operator $\nabla = (\partial_1, \partial_2, \partial_3)$ with the vectorfield $\mathbf{V} = (v_1, v_2, v_3)$, but you need to remember that $\partial_i v_j$ is the partial derivative of the function v_j with respect to coordinate x_i and not multiplication, so issues related to the Leibniz product rule may arise. It may also be helpful to think each of the six vectorfields $x_i\mathbf{e}_j$ (for $1 \le i \ne j \le 3$) carries "quantum" or "bit" or "qubit" of **curl**, keeping in mind that this bit is *vector-valued*: for example, $\mathbf{curl}(x_1\mathbf{e}_2) = \mathbf{e}_3$ and $\mathbf{curl}(x_2\mathbf{e}_1) = -\mathbf{e}_3$.

Problem 0.1. *Compute* $\mathbf{curl}(\mathbf{V}) = \nabla \times \mathbf{V}$ *for the following vectorfields on* $\mathbf{R}^3$:

- $\mathbf{V} = (x_2x_3, x_1x_3, x_1x_2) = x_2x_3\mathbf{e}_1 + x_1x_3\mathbf{e}_2 + x_1x_2\mathbf{e}_3$;
- $\mathbf{V} = (2x_3, 5x_1, 4x_2) = 2x_3\mathbf{e}_1 + 5x_1\mathbf{e}_2 + 4x_2\mathbf{e}_3$;
- $\mathbf{V} = \frac{1}{2}(a, b, c) \times (x_1, x_2, x_3) = \frac{1}{2}\mathbf{A} \times \mathbf{x}$, *where* $\mathbf{A} = (a, b, c) = a\mathbf{e}_1 + b\mathbf{e}_2 + c\mathbf{e}_3$ *is any* constant *vectorfield.*

Problem 0.2. *Show that* $\mathbf{curl}(\mathbf{grad}(f)) = \nabla \times \nabla f = \mathbf{0}$ *for any smooth function* f *on* $R \subset \mathbf{R}^3$.

Problem 0.3. *Show that* $\mathbf{div}(\mathbf{curl}(\mathbf{V})) = \nabla \cdot (\nabla \times \mathbf{V}) = 0$ *for any smooth vectorfield* $\mathbf{V}$ *on* $R \subset \mathbf{R}^3$.

Problem 0.4. *Let* f *be any smooth function, and let* $\mathbf{U}, \mathbf{V}$ *be any smooth vectorfields, on* $R \subset \mathbf{R}^3$. *Work out Leibniz product formulae for*

- $\mathbf{curl}(f\mathbf{U}) = \nabla \times (f\mathbf{U})$,
- $\mathbf{curl}(\mathbf{U} \times \mathbf{V})) = \nabla \times (\mathbf{U} \times \mathbf{V})$,

as well as a formula for

- $\mathbf{curl}(\mathbf{curl}(\mathbf{V})) = \nabla \times (\nabla \times \mathbf{V})$.

[Hint: The "BAC-CAB" formula $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$ — so called, because the parentheses go "in the back"— can serve as a guide to the last two formulae!]

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