PRACTICE WITH PATH INTEGRALS

Recall that a *path* $\mathcal{P}$ is a continuous function (a *map* or *parametrization*) $\mathbf{X} : [a, b] \to R \subset \mathbf{R}^n$ from an interval $[a, b] \subset \mathbf{R}$ into a region R of n-space, and a *loop* is a path with $\mathbf{X}(a) = \mathbf{X}(b)$. Often we take the interval to be $[0, 1]$, and we normally assume the path $\mathcal{P}$ is C^1, that is, the velocity $\mathbf{X}'(t) = (x_1'(t), \dots, x_n'(t))$ of its parametrization is a continuous $\mathbf{R}^n$-valued function of $t \in [0, 1]$. Sometimes we will abuse notation, conflating the path (which is parametrized and thus *oriented*) with its image and writing $\mathcal{P} = \{\mathbf{X}(t) \,|\, t \in [0, 1]\}$.

The *path integral* along $\mathcal{P}$ is defined by

$$\int_{\mathcal{P}} \mathbf{V} \cdot d\mathbf{X} := \int_0^1 \mathbf{V}(\mathbf{X}(t)) \cdot \mathbf{X}'(t)\, dt = \int_0^1 (v_1(\mathbf{X}(t))x_1'(t) + \cdots + v_n(\mathbf{X}(t))x_n'(t))\, dt$$

for any vectorfield $\mathbf{V} = \mathbf{V}(\mathbf{X}) = (v_1(x_1, , \dots, x_n), \dots, v_n(x_1, , \dots, x_n))$ on $R \subset \mathbf{R}^n$.

Consider these paths in $\mathbf{R}^3$

$$\mathcal{L} := \{\mathbf{X}(t) := (4t, 2t, 5t) \,|\, t \in [0, 1]\} \text{ (a line segment)}$$
$$\mathcal{T} := \{\mathbf{X}(t) := (4t, 2t^2, 5t^3) \,|\, t \in [0, 1]\} \text{ (part of a twisted cubic curve)}$$
$$\mathcal{H} := \{\mathbf{X}(t) := (\cos t, \sin t, t) \,|\, t \in [0, 2\pi]\} \text{ (part of helix - note the interval)}$$
$$\mathcal{S} := \{\mathbf{X}(t) := (\cos t, \sin t, 0) \,|\, t \in [0, 2\pi]\} = S^1 \text{ (the unit circle - again, note the interval)}$$

and these vectorfields on (a region of) $\mathbf{R}^3$

$$\mathbf{U} := (1, 0, 0) = \mathbf{e}_1$$
$$\mathbf{V} := (x_2x_3, x_1x_3, x_1x_2) = x_2x_3\mathbf{e}_1 + x_1x_3\mathbf{e}_2 + x_1x_2\mathbf{e}_3$$
$$\mathbf{W} := (-x_2, x_1, 0) = -x_2\mathbf{e}_1 + x_1\mathbf{e}_2$$
$$\mathbf{T} := (-x_2, x_1, 0)/(x_1^2 + x_2^2) = \mathbf{W}/\rho^2 = \nabla\theta$$

Problem 0.1. *Compute* $\int_{\mathcal{L}} \mathbf{U} \cdot d\mathbf{X}$ *and* $\int_{\mathcal{T}} \mathbf{U} \cdot d\mathbf{X}$. *Are your answers the same? Why or why not?* [Hint: Is $\mathbf{U}$ the gradient of a function?]

Problem 0.2. *Compute* $\int_{\mathcal{L}} \mathbf{V} \cdot d\mathbf{X}$ *and* $\int_{\mathcal{T}} \mathbf{V} \cdot d\mathbf{X}$. *Are your answers the same? Why or why not?*

Problem 0.3. *Compute* $\int_{\mathcal{H}} \mathbf{U} \cdot d\mathbf{X}$ *and* $\int_{\mathcal{S}} \mathbf{U} \cdot d\mathbf{X}$. *Are your answers the same? Why or why not?*

Problem 0.4. *Compute* $\int_{\mathcal{H}} \mathbf{W} \cdot d\mathbf{X}$ *and* $\int_{\mathcal{S}} \mathbf{W} \cdot d\mathbf{X}$. *Are your answers the same? Why or why not?*

Problem 0.5. *Compute* $\int_{\mathcal{H}} \mathbf{T} \cdot d\mathbf{X}$ *and* $\int_{\mathcal{S}} \mathbf{T} \cdot d\mathbf{X}$. *Are your answers the same? Why or why not?*

Problem 0.6 (Extra Credit)**.** *What happens to your answers for the previous three problems if you reparametrize the* helix *by* $\mathcal{H} := \{\mathbf{X}(t) := (\cos 2\pi t, \sin 2\pi t, 2\pi t) \,|\, t \in [0, 1]\}$ *or the* unit circle *by* $\mathcal{S} := \{\mathbf{X}(t) := (\cos 2\pi t, \sin 2\pi t, 0) \,|\, t \in [0, 1]\}$*? Explain!*

Rob Kusner, Mathematics & Statistics, University of Massachusetts, Amherst, MA 01003, USA
E-mail address: profkusner@gmail.com

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