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Curling...

Recall the curl of a vector field

$$\vec{v} = (v_1, v_2, v_3) = v_1\hat{e}_1 + v_2\hat{e}_2 + v_3\hat{e}_3$$

on $\mathbb{R}^3$ is the vector field

$$\boxed{\begin{aligned} \operatorname{curl}\vec{v} = \nabla\times\vec{v} &= (\partial_1, \partial_2, \partial_3)\times(v_1, v_2, v_3) \\ &:= (\partial_2 v_3 - \partial_3 v_2,\ \partial_3 v_1 - \partial_1 v_3,\ \partial_1 v_2 - \partial_2 v_1) \\ &:= \hat{e}_1(\partial_2 v_3 - \partial_3 v_2) + \hat{e}_2(\partial_3 v_1 - \partial_1 v_3) + \hat{e}_3(\partial_1 v_2 - \partial_2 v_1) \end{aligned}}$$

Compute the curl $\nabla\times\vec{v}$ of the following vector fields on $\mathbb{R}^3$:

1. $\vec{v} = x_1\hat{e}_1 + x_2\hat{e}_2 = (x_1, x_2, 0)$

2. $\vec{v} = x_1\hat{e}_1 - x_2\hat{e}_2 = (x_1, -x_2, 0)$

3. $\vec{v} = x_2\hat{e}_1 - x_1\hat{e}_2 = (x_2, -x_1, 0)$

(You might also sketch some pictures to help you interpret the curl geometrically.)

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The remaining 3 problems are general facts & formulas for <u>curl</u>

4. Show $\boxed{\text{curl}(\text{grad} f) = \nabla \times \nabla f = 0}$ for any function f on $\mathbb{R}^3$.

5. Show $\boxed{\text{div}(\text{curl}\, \vec{v}) = \nabla \cdot (\nabla \times \vec{v}) = 0}$ for any vector field $\vec{v}$ on $\mathbb{R}^3$.

6. Work out a formula for $\boxed{\text{curl}(\text{curl}\, \vec{v}) = \nabla \times (\nabla \times \vec{v})}$

in terms of $\text{grad}(\text{div}\, \vec{v}) = \nabla(\nabla \cdot \vec{v})$

and "$\text{div}(\text{grad}\, \vec{v})$" $= (\nabla \cdot \nabla)\vec{v}$

$$= (\nabla \cdot \nabla v_1, \nabla \cdot \nabla v_2, \nabla \cdot \nabla v_3)$$
$$= (\Delta v_1, \Delta v_2, \Delta v_3)$$
$$= \Delta \vec{v}$$

for any vector field $\vec{v}$ on $\mathbb{R}^3$.