425g. pdf Fun with Forms Rob Kusner

(see, e.g. M&T § 8.6)

Recall the basic 1-forms on $\mathbb{R}^3$ are dx, dy, dz, and any 1-form on $\mathbb{R}^3$ is a combination

$$\boxed{\alpha = \alpha_1 dx + \alpha_2 dy + \alpha_3 dz}$$

where $\alpha_i = \alpha_i(x,y,z)$ is a function on $\mathbb{R}^3$ for $i = 1,2,3$.

If $f = f(x,y,z)$ is a function (or 0-form) on $\mathbb{R}^3$, we define its exterior derivative (or "total derivative")

$$\boxed{df = f_x\, dx + f_y\, dy + f_z\, dz}$$

which is a 1-form (indeed, an exact 1-form).

We can make higher degree forms by exterior product:

for example, the basic 2-forms on $\mathbb{R}^3$ are

$dx \wedge dy = -dy \wedge dx$, $dy \wedge dz = -dz \wedge dy$, $dz \wedge dx = -dx \wedge dz$

(note that $dx \wedge dx = dy \wedge dy = dz \wedge dz = 0$ why?) and we extend linearly to multiply any pair of 1-forms α and $\tilde{\alpha} = \tilde{\alpha}_1 dx + \tilde{\alpha}_2 dy + \tilde{\alpha}_3 dz$ as follows:

$$\boxed{\begin{aligned}\alpha \wedge \tilde{\alpha} &= (\alpha_1 dx + \alpha_2 dy + \alpha_3 dz) \wedge (\tilde{\alpha}_1 dx + \tilde{\alpha}_2 dy + \tilde{\alpha}_3 dz) = \\ &(\alpha_1\tilde{\alpha}_2 - \alpha_2\tilde{\alpha}_1)\, dx \wedge dy + (\alpha_2\tilde{\alpha}_3 - \alpha_3\tilde{\alpha}_2)\, dy \wedge dz + (\alpha_3\tilde{\alpha}_1 - \alpha_1\tilde{\alpha}_3)\, dz \wedge dx\end{aligned}}$$

– this "wedge product" of 1-forms should remind you of cross product!

We can also apply exterior derivative to 1-forms, giving the 2-form:

$$\boxed{\begin{aligned} d\alpha &= d\alpha_1 \wedge dx + d\alpha_2 \wedge dy + d\alpha_3 \wedge dz \\ &= ((\alpha_2)_x - (\alpha_1)_y)\, dx \wedge dy + ((\alpha_3)_y - (\alpha_2)_z)\, dy \wedge dz + ((\alpha_1)_z - (\alpha_3)_x)\, dz \wedge dx \end{aligned}}$$

using the skew symmetry of wedge product and the fact that $d(dx) = d(dy) = d(dz) = 0$. This formula should remind you of <u>curl</u>!

And if $\beta = \beta_{12}\, dx \wedge dy + \beta_{23}\, dy \wedge dz + \beta_{31}\, dz \wedge dx$ is a general 2-form, we can take its exterior derivative

$$d\beta = ((\beta_{12})_z + (\beta_{23})_x + (\beta_{31})_y)\, dx \wedge dy \wedge dz$$

which should remind you of <u>div</u>!

(1) What vector field does df remind you of? Why?

(2) Show that $d(df) = 0$ for any function f.

(3) Is the 1-form $\alpha = x\,dy + dz$ exact, i.e. is there a function (0-form) f with $\alpha = df$?

(4) Show $d(d\alpha) = 0$ for any 1-form α.

(5) Show $d(\alpha \wedge \tilde{\alpha}) = d\alpha \wedge \tilde{\alpha} - \alpha \wedge d\tilde{\alpha}$ for any 1-forms α & β.

(6) Compute $d(d\beta)$ for any 2-form β.