Linear Algebra Final Exam Practice Problems

Spring 2011 Math 313/513 at Penn

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1: Let V, W be vector spaces. Define the following terms:

1a: What is a subspace of V?

1b: Let $F : V \to W$ be a function. What does it mean to say that F is linear?

1c: Let $S = \{v_1, v_2, \cdots\}$ be a subset of V. What is a linear combination of elements of S? What is the span of S? What does it mean to say that S is linearly independent? What does it mean to say that S spans V? What does it mean to say that S is a basis of V?

1d: What is the dimension of V?

1e: Let $F : V \to W$ be linear. Define $ker(F)$. Define $im(F)$. What is the rank of F? What is the nullity of F?

1f: Let $F : V \to V$ be linear. What is an eigenvalue of F? What is an eigenvector of F?

1g: What does it means to say that two $n \times n$ matrices are similar?

1h: What does it mean to say that two vector spaces are isomorphic?

1i: Let A be an $n \times n$ matrix. What is an eigenbasis for the matrix A?

1k: Let B be a basis of a vector space V. What does one mean by the coordinates of a vector $v \in V$ with respect to B?

2a: Let $F : V \to W$ be linear. Show that $ker(F)$ is a subspace of V. Show that $im(F)$ is subspace of W.

2b: State the rank+nullity theorem.

2c: When $V = W$ has finite dimension n, explain why F is an isomorphism iff nullity$= 0$ iff rank$= n$.

2d: Give an example where 2c fails for infinite dimensional V.

3: Consider the system of equations

$$\begin{aligned} x - 2y + 3z - w &= 2 \\ 2x + y - z + 3w &= 1 \\ 5x + z + 5w &= 4. \end{aligned}$$

3a: Write the system as a matrix equation.

3b: Find all solutions to this system, if any, and describe these geometrically.

4a: Which vectors $X = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ are in the image of the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 \\ -2 & 1 & -5 \\ 3 & -2 & 8 \end{pmatrix}?$$

4b: Which $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ are a linear combination of $v_1 = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}, v_3 = \begin{pmatrix} 2 \\ -5 \\ 8 \end{pmatrix}$?

4c: What is the orthogonal projection of $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ to the span of v_1, v_2, v_3? (Use Gram-Schmidt to find an orthonormal basis, or try some shortcut.)

4d: What is the least-squares solution to the system $AX = e_1$?

5: Let A denote the matrix representing rotation by angle $\pi/6$ about the line through the origin and the point $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$. Let B be the matrix representing reflection across the plane $3x - y + z = 0$.
Explain how you find the matrix representing the composition of the reflection followed by the rotation from the matrices A and B? (You are not asked to find the matrices A and B or the matrix representing the composition — that takes some work!)

6: Let $A = \{\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}\}$ be a basis of $\mathbb{R}^2$. Let $M = \begin{pmatrix} 1 & -2 \\ 3 & 0 \end{pmatrix}$ be the matrix representing a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ with respect to the basis $E = \{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$. What is the representation this linear transformation with respect to the basis A?

7: True or False. (Explain!)

7a: The set of all vectors of the form $\begin{pmatrix} a \\ b \\ 0 \\ a \end{pmatrix}$ where a, b are real numbers forms a subspace.

7b: Let V be the space of all functions from $\mathbb{R}$ to $\mathbb{R}$ that have infinitely many derivatives. The function

$$F : V \to V$$
$$F : f \mapsto 3f' - 2f''$$

is linear.

7c: If the determinant of a 4×4 matrix is 4, then the rank of the matrix must be 4.

7d: If the standard vectors $\{e_1, e_2, \cdots e_n\}$ are eigenvectors of an $n \times n$ matrix, then the matrix is diagonal.

7e: If 1 is the only eigenvalue of an $n \times n$ matrix A, then A must be I_n.

7f: If two 3×3 matrices both have the eigenvalues $3, 4, 5$, then A must be similar to B.

8a: Let F be counterclockwise rotation of the plane by angle 60 degrees followed by a scaling of $\frac{3}{2}$. What are all the eigenvalues of F.

8b: What are the eigenvalues and eigenvectors of orthogonal projection onto a line L in $\mathbb{R}^3$?

9: Let A be a 2×2 matrix with eigenvlaues $1.3, .6.$ and corresponding eigenvectors $\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. Let $v = \begin{pmatrix} 15 \\ 15 \end{pmatrix}$. Find $A^n(v)$ for $n = 61$. Your answer will have expressions of the form $(.6)^p, (1.3)^p$. Do not simplify these.

10a: Find the eigenvalues and eigenvectors for the matrix

$$A = \begin{pmatrix} 4 & 2 \\ 2 & 7 \end{pmatrix}.$$

Find a matrix B so that

$$BAB^{-1} = M$$

is diagonal. What is the matrix M.

10b: Also find the polar (or QR) decomposition for A and compare with 10a.

10c: Find the eigenvalues and eigenvectors for the matrix

$$\begin{pmatrix} 8 & 9 \\ -4 & -4 \end{pmatrix}.$$

Is this matrix diagonalizable. If it is what is the diagonal matrix? If not diagonalizable, why not?

11: Find the eigenvalues of the matrix A, given below. Find bases for the eigenspaces of A. Can you find an invertible matrix, S, such that $S^{-1}AS = D$, where D is a diagonal matrix? If no, why not? If yes, find the matrices S and D.

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 3 & 2 \end{pmatrix}.$$

Hint: When computing the characteristic polynomial of A, watch out for common factors: you want it factored at the end of the day.

12: Find the eigenvalues of the matrix A, given below. Find bases for the eigenspaces of A. Can you find an invertible matrix, S, such that $S^{-1}AS = D$, where D is a diagonal matrix? If no, why not? If yes, find the matrices S and D.

$$A = \begin{pmatrix} -8 & 5 & 4 \\ -9 & 5 & 5 \\ 0 & 1 & 0 \end{pmatrix}.$$

Hint: One way to solve a cubic equation is to guess a root, then perform long division, which would leave you with a quadratic polynomial. If the characteristic polynomial has a free coefficient which is an integer, as a first guess you may want to check the numbers which divide it. For example, if you have $\lambda^3 - 2\lambda^2 - \lambda + 2$, you may want to try ± 1 and ± 2.

13: Find the eigenvalues of the matrix A, given below. Find bases for the eigenspaces of A. Can you find an invertible matrix, S, such that $S^{-1}AS = D$, where D is a diagonal matrix? If no, why not? If yes, find the matrices S and D.

$$A = \begin{pmatrix} 3 & 2 & -2 \\ 2 & 3 & -2 \\ 6 & 6 & -5 \end{pmatrix}.$$

14: Find the determinant of the matrix

$$\begin{pmatrix} -1 & 2 & 0 \\ 2 & -2 & 5 \\ 4 & -1 & 3 \end{pmatrix}$$

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