Practice Midterm Problems for Math 313/513 at Penn, Spring 2011

1. a: Write the system of equations below as a matrix equation:

$$x + y - z = 6$$

$$2x - y = 0$$

$$3x - y - 2z = -3$$

b: What is the LU factorization of the coefficient matrix?
c: Solve the system of equations using row operations.

2. a: For what vectors $v = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ does the equation $Ax = v$ have a solution if $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 3 \end{pmatrix}$, and $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$.
b: What is the rank of the matrix A?
c: What is the dimension of $\mathrm{im}(A)$? What is the dimension of $\ker(A)$?

3. What does it mean for a vector to be in the image of a matrix A.
Let A be the matrix $\begin{pmatrix} 1 & 2 & 5 \\ -2 & 0 & -2 \\ 3 & -1 & 1 \end{pmatrix}$. Is $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ an element of the image of A?
Why or why not?

4. a: Define what it means for a set S to be a basis of a vector space V.
b: Let

$$A = \begin{pmatrix} 1 & 2 & 3 & -1 \\ -1 & 0 & 1 & -1 \\ -1 & 4 & 3 & -5 \end{pmatrix}.$$

Give a set of vectors that span $\mathrm{im}(A)$ and that are independent.
c: Find a basis for the image of the linear map from $\mathbb{R}^{2\times 2}$ to itself that sends any 2×2 matrix A to the matrix $A - A^T$. (Here and below A^T denotes the transpose of the matrix A.)

5. Let A be a $p \times q$ matrix, so A gives a linear map from $\mathbb{R}^q$ to $\mathbb{R}^p$. Let $X_1, X_2 \in \mathbb{R}^q$. Assume that $A(X_1) = 0$ and that $A(X_2) = B$. Explain why $A(X_1 + X_2) = B$, and conversely, why every solution to $A(X) = B$ has the form $X = X_1 + X_2$.

6. Let A be a two by two matrix that rotates by angle $\pi/4$.
a: Find A.
b: Give a geometric explanation for why $A^{24} = I_2$. (Here I_2 denotes the 2 by 2 identity matrix.) What other powers of A yield I_2? What powers of A yield $-I_2$?

7. Solve the equation

$$\begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 2 \\ 2 & 1 & -1 \end{pmatrix} X = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}.$$

for $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ by first finding the inverse of the given coefficient matrix.

8. Compute the products AB and BA of the two matrices A, B given below, and compare their ranks:

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 0 \\ 3 & -2 \end{pmatrix}$$

$$B = \begin{pmatrix} -1 & 0 & 0 \\ 4 & 8 & 7 \end{pmatrix}$$

9. Let W denote the set of vectors in $\mathbb{R}^3$ orthogonal to the vector $\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$. Argue that W is a subspace and find a basis of W.

10. Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 2 & 0 \\ 1 & 1 & 3 \\ -2 & 1 & -3 \end{pmatrix}.$$

Let $B = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$. Find equations in b_1, b_2, b_3, b_4 so that the equation $AX = B$ can be solved. Find a basis of the image of A.

11. True or False. (Please give a reason if True or a counterexample if False.)
a: The image of a 3×4 matrix A is a subspace of $\mathbb{R}^4$.

b: If the kernel of a matrix A consists of the zero vector only, then the column vectors of A must be linearly independent.

c: If u, v, w are in a subspace of any vector space, then so is $3u - v - w$.

d: The function

$$T : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x - y \\ x + y \end{pmatrix}$$

is a linear map.

e: If A is an invertible $n \times n$ matrix, then the reduced row echelon form of A is I_n.

f: The formula $AB = BA$ holds for all $n \times n$ matrices A and B.

g: If the 4×4 matrix A has rank 4, then any linear system with coefficient matrix A will have a unique solution.

12. An $n \times n$ matrix M defines a linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$. Note that M^2 also defines a linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$.
a: Show that if $v \in \operatorname{im}(M^2)$, then $v \in \operatorname{im}(M)$, meaning we have the inclusion of subspaces $\operatorname{im}(M^2) \subset \operatorname{im}(M)$.
b: What are the corresponding statements about $\ker(M)$ and $\ker(M^2)$?
c: Find a 3×3 matrix M with $\operatorname{im}(M) \neq \operatorname{im}(M^2)$ and also (necessarily – why?!) $\ker(M) \neq \ker(M^2)$.

13. Find the 3×3 matrix that represents rotation of $\mathbb{R}^3$ by angle $2\pi/3$ about the line spanned by $(1, 1, 1)$. Explain why this is a permutation matrix.

14. Let $f : \mathbb{R}^3 \to \mathbb{R}^5$ be a linear map. We are given that

$$f(u) = (0, 1, 2, 3, 4)$$
$$f(v) = (-1, 2, 6, 1, 4).$$

What is $f(2u - 3v)$?

15. Which of the following are subspaces of the indicated space. Explain your answer.
a: The set of of solutions in $\mathbb{R}^3$ to the equation $3x - y + 2z = 1$.
b: The set of vectors in $\mathbb{R}^4$ orthogonal to $(1, 2, 3, -4)$.
c: The subset of all 3×3 matrices A satisfying $A^T = 2A$.

16. The matrix M of size 4×6 has kernel with dimension 2. How many independent column vectors does M have? Why? What is the dimension of the image of M? Why ?

17. Let

$$A = \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}, B = \begin{pmatrix} -1 & -4 \\ 4 & 7 \end{pmatrix}.$$

Find a 2×2 matrix X so that

$$AX = B.$$