1. Let $V, W$ be vector spaces and $F : V \to W$ be a linear map.

(a) Define $\ker(F)$.

(b) Define the rank and nullity of $F$.

(a) The kernel of $F$ is given by

$$\ker(F) = \{ x \in V \mid F(x) = 0 \}.$$  

In words, the kernel of $F$ is the set of all elements $x \in V$ such that $F(x) = 0$.

(b) The rank of $F$ is the dimension of the image of $F$. (The image of $F$ is given by

$$\text{im}(F) = \{ y \in W \mid y = F(x) \text{ for some } x \in V \}.$$  

In words, the image of $F$ is the set of all elements $y \in W$ such that $y = F(x)$ for some $x \in V$. The nullity of $F$ is the dimension of the kernel of $F$.

2. Let $F : \mathbb{R}^5 \to \mathbb{R}^3$ be the linear map given by the matrix

$$A = \begin{pmatrix} 2 & -1 & 1 & 0 & 2 \\ 2 & -1 & -1 & 1 & 0 \\ 6 & -3 & -1 & -2 & 2 \end{pmatrix}$$

Find a basis of the kernel of $F$ and a basis of the image of $F$. What is the rank of $F$?
We apply the row reduction algorithm to the above matrix $A$ to obtain the row reduced echelon form

$$B = \begin{pmatrix}
1 & -\frac{1}{2} & 0 & 0 & \frac{1}{2} \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix}$$

The image of $F$ is spanned by the columns of $A$. The columns of $A$ corresponding to pivot columns of $B$ give a basis of $\text{im}(F)$. So in our example

$$\begin{pmatrix}
2 \\
2 \\
6
\end{pmatrix}, \begin{pmatrix}
1 \\
-1 \\
-1
\end{pmatrix}, \begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix}$$

is a basis of $\text{im}(F)$. (Alternatively, note that $\text{im}(F) = \mathbb{R}^3$, so a basis is given by the standard basis

$$\begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}, \begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix}, \begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix}$$

of $\mathbb{R}^3$ (for example).) A basis for the kernel is obtained by solving the system of linear equations $Ax = 0$ using the rref $B$ of $A$ (the Gaussian elimination algorithm). The non-pivot columns of $B$ correspond to free variables (in this case $x_2$ and $x_5$), and the rows of $B$ give equations which can be solved for the remaining variables in terms of the free variables:

$$x_1 = \frac{1}{2}x_2 - \frac{1}{2}x_5$$
$$x_3 = -x_5$$
$$x_4 = 0$$

So if $Ax = 0$ then

$$x = \begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{pmatrix} = \begin{pmatrix}
\frac{1}{2}x_2 - \frac{1}{2}x_5 \\
x_2 \\
-x_5 \\
0 \\
x_5
\end{pmatrix} = x_2 \begin{pmatrix}
\frac{1}{2} \\
1 \\
0 \\
0 \\
0
\end{pmatrix} + x_5 \begin{pmatrix}
-\frac{1}{2} \\
0 \\
1 \\
0 \\
0
\end{pmatrix}$$

Now a basis of the kernel is given by

$$\begin{pmatrix}
\frac{1}{2} \\
1 \\
0 \\
0 \\
0
\end{pmatrix}, \begin{pmatrix}
-\frac{1}{2} \\
0 \\
-1 \\
0 \\
1
\end{pmatrix}.$$
Finally, the rank of $F$ equals the dimension of the image of $F$ (that is, the number of vectors in a basis of the image of $F$), so $\text{rank}(F) = 3$.

(Please note that if the entry $-2$ in the last row of the original matrix were replaced with a 2, the resulting matrix $F'$ has $\text{rank}(F') = 2$ and several features of this problem change.)

3. Let $\mathcal{A} = \left\{ \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix} \right\}$ and $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ be two bases of $\mathbb{R}^2$.

(a) For a vector $v \in \mathbb{R}^2$ let $[v]_{\mathcal{A}}$ denote its $\mathcal{A}$-coordinate vector and $[v]_{\mathcal{B}}$ its $\mathcal{B}$-coordinate vector. Find a matrix $[1]_{\mathcal{B},\mathcal{A}}$ such that

$$[1]_{\mathcal{B},\mathcal{A}}[v]_{\mathcal{A}} = [v]_{\mathcal{B}}.$$ 

(b) Find a matrix $[1]_{\mathcal{A},\mathcal{B}}$ such that $[1]_{\mathcal{A},\mathcal{B}}[v]_{\mathcal{B}} = [v]_{\mathcal{A}}$.

(c) Let $M = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$ be the $\mathcal{B}$-matrix of a linear map $T: \mathbb{R}^2 \to \mathbb{R}^2$.

What is the $\mathcal{A}$-matrix $N = [T]_{\mathcal{A},\mathcal{A}}$ of $T$?

(a) $$[1]_{\mathcal{B},\mathcal{A}} = \left( \begin{pmatrix} 2 \\ 1 \end{pmatrix}_{\mathcal{B}} \begin{pmatrix} 3 \\ 2 \end{pmatrix}_{\mathcal{B}} \right) = \begin{pmatrix} 2 \\ 3 \\ 1 \\ 2 \end{pmatrix}.$$ 

(b) $$[1]_{\mathcal{A},\mathcal{B}} = [1]_{\mathcal{B},\mathcal{A}}^{-1} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}^{-1} = \frac{1}{2 \cdot 2 - 3 \cdot 1} \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}.$$ 

(c) $$N = [1]_{\mathcal{A},\mathcal{B}} M [1]_{\mathcal{B},\mathcal{A}} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -19 \\ 13 \\ -34 \\ 23 \end{pmatrix}.$$ 

4. True or False. You must explain your answer.

(a) Let $M$ be a matrix. If the kernel of $M$ equals $\{0\}$, then the columns of $M$ are linearly independent.

(b) If $V$ is a subspace of $\mathbb{R}^n$ and $u, v, w \in V$, then $2u - 3v + 4w \in V$ also.
(a) True. Let \( v_1, \ldots, v_n \) be the columns of \( M \) and let

\[
x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}
\]

be a vector in \( \mathbb{R}^n \). Then

\[
Mx = x_1v_1 + \cdots + x_nv_n
\]

is the linear combination of the columns of \( M \) with coefficients the entries \( x_i \) of \( x \). So, if \( \ker(M) = \{0\} \), then

\[
x_1v_1 + \cdots + x_nv_n = 0 \Rightarrow x_1 = \cdots = x_n = 0.
\]

That is, \( v_1, \ldots, v_n \) are linearly independent.

(b) True. A subset \( V \subset \mathbb{R}^n \) is a subspace if \( 0 \in V \) and \( V \) is closed under addition and scalar multiplication. In particular, any linear combination of elements of \( V \) (such as \( 2u - 3v + 4w \)) is also an element of \( V \).

5. Let \( S \) be the set of \( 2 \times 2 \) matrices \( A \) such that

\[
A \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\]

Find a basis for \( S \). What is the dimension of \( S \)?

Write

\[
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
\]

Then

\[
A \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} a + b & a + b \\ c + d & c + d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\]

gives \( a + b = 0 \) and \( c + d = 0 \). So \( b = -a \) and \( d = -c \), and we have

\[
S = \left\{ \begin{pmatrix} a & -a \\ c & -c \end{pmatrix} \mid a, c \in \mathbb{R} \right\} = \text{Span} \left( \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \right).
\]

We see that

\[
\left\{ \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \right\}
\]
is a basis of \( S \) (it spans \( S \) and is linearly independent). The dimension of a vector space is the size of a basis, so the dimension of \( S \) equals 2.

6. Let \( \mathcal{P}_2 \) denote the vector space of polynomials of degree less than or equal to 2. Find a basis for the image and kernel of the linear map

\[ T: \mathcal{P}_2 \to \mathbb{R}^2, \quad T(f) = \begin{pmatrix} f(1) \\ f'(1) \end{pmatrix}. \]

Write \( f(t) = a_0 + a_1 t + a_2 t^2 \). Then \( f'(t) = a_1 + 2a_2 t \). So

\[ T(f) = \begin{pmatrix} f(1) \\ f'(1) \end{pmatrix} = \begin{pmatrix} a_0 + a_1 + a_2 \\ a_1 + 2a_2 \end{pmatrix}. \]

A basis of the kernel is given by solving the system of linear equations

\[
\begin{align*}
    a_0 + a_1 + a_2 &= 0 \\
    a_1 + 2a_2 &= 0
\end{align*}
\]

We find (by Gaussian elimination, or inspection)

\[
\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = a_2 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.
\]

So the kernel of \( T \) has basis \( \{1 - 2t + t^2\} \) given by the polynomial corresponding to the vector

\[
\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.
\]

By the rank-nullity formula, the image of \( T \) has dimension 2, so \( \text{im}(T) = \mathbb{R}^2 \), and we can take the standard basis

\[
\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}.
\]

7. Let \( \mathcal{A} \) be the basis \( \{1, t, t^2\} \) of \( \mathcal{P}_2 \), the set of all polynomials of degree less than or equal to 2. Find the \( \mathcal{A} \)-matrix of the linear map

\[ T: \mathcal{P}_2 \to \mathcal{P}_2, \quad T(f) = tf' + 2f'' - f. \]
Write \( f(t) = a_0 + a_1 t + a_2 t^2 \). Then \( f'(t) = a_1 + 2a_2 t \) and \( f''(t) = 2a_2 \), so

\[
T(f) = t(a_1 + 2a_2 t + 2 \cdot (2a_2) - (a_0 + a_1 t + a_2 t^2) = (4a_2 - a_0) + a_2 t^2 = a_0(-1) + a_1(0) + a_2(4 + t^2).
\]

So the \( A \)-matrix of \( T \) is the matrix

\[
[T]_A = \begin{pmatrix}
-1 & 0 & 4 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

whose columns are the \( A \)-coordinate vectors of the polynomials \( T(1), T(t), T(t^2) \) (these are the polynomials \(-1, 0, 4 + t^2\) occurring with coefficients \( a_0, a_1, a_2 \) in the expression for \( T(f) \) above).