Answers for Maxi-Midterm

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1. Let V, W be vector spaces and $F\colon V \to W$ be a linear map.

(a) Define $\ker(F)$.

(b) Define the rank and nullity of F.

(a) The kernel of F is given by

$$\ker(F) = \{\mathbf{x} \in V \mid F(\mathbf{x}) = \mathbf{0}\}.$$

In words, the kernel of F is the set of all elements $\mathbf{x} \in V$ such that $F(\mathbf{x}) = \mathbf{0}$.

(b) The rank of F is the dimension of the image of F. (The image of F is given by

$$\operatorname{im}(F) = \{\mathbf{y} \in W \mid \mathbf{y} = F(\mathbf{x}) \text{ for some } \mathbf{x} \in V\}.$$

In words, the image of F is the set of all elements $\mathbf{y} \in W$ such that $\mathbf{y} = F(\mathbf{x})$ for some $\mathbf{x} \in V$.) The nullity of F is the dimension of the kernel of F.

2. Let $F\colon \mathbb{R}^5 \to \mathbb{R}^3$ be the linear map given by the matrix

$$A = \begin{pmatrix} 2 & -1 & 1 & 0 & 2 \\ 2 & -1 & -1 & 1 & 0 \\ 6 & -3 & -1 & -2 & 2 \end{pmatrix}$$

Find a basis of the kernel of F and a basis of the image of F. What is the rank of F?

We apply the row reduction algorithm to the above matrix A to obtain the row reduced echelon form

$$B = \begin{pmatrix} 1 & -\frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The image of F is spanned by the columns of A. The columns of A corresponding to pivot columns of B give a basis of $\mathrm{im}(F)$. So in our example

$$\left\{ \begin{pmatrix} 2 \\ 2 \\ 6 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \right\}$$

is a basis of $\mathrm{im}(F)$. (Alternatively, note that $\mathrm{im}(F) = \mathbb{R}^3$, so a basis is given by the standard basis

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

of $\mathbb{R}^3$ (for example).) A basis for the kernel is obtained by solving the system of linear equations $A\mathbf{x} = \mathbf{0}$ using the rref B of A (the Gaussian elimination algorithm). The non-pivot columns of B correspond to free variables (in this case x_2 and x_5), and the rows of B give equations which can be solved for the remaining variables in terms of the free variables:

$$\begin{aligned} x_1 &= \tfrac{1}{2}x_2 - \tfrac{1}{2}x_5 \\ x_3 &= -x_5 \\ x_4 &= 0 \end{aligned}$$

So if $A\mathbf{x} = \mathbf{0}$ then

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}x_2 - \frac{1}{2}x_5 \\ x_2 \\ -x_5 \\ 0 \\ x_5 \end{pmatrix} = x_2 \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -\frac{1}{2} \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

Now a basis of the kernel is given by

$$\left\{ \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Finally, the rank of F equals the dimension of the image of F (that is, the number of vectors in a basis of the image of F), so $\operatorname{rank}(F) = 3$.
(Please note that if the entry -2 in the last row of the original matrix were replaced with a 2, the resulting matrix F' has $\operatorname{rank}(F') = 2$ and several features of this problem change.)

3. Let $\mathcal{A} = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix} \right\}$ and $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ be two bases of $\mathbb{R}^2$.

(a) For a vector $\mathbf{v} \in \mathbb{R}^2$ let $[\mathbf{v}]_{\mathcal{A}}$ denote its $\mathcal{A}$-coordinate vector and $[\mathbf{v}]_{\mathcal{B}}$ its $\mathcal{B}$-coordinate vector. Find a matrix $[1]_{\mathcal{BA}}$ such that

$$[1]_{\mathcal{BA}}[\mathbf{v}]_{\mathcal{A}} = [\mathbf{v}]_{\mathcal{B}}.$$

(b) Find a matrix $[1]_{\mathcal{AB}}$ such that $[1]_{\mathcal{AB}}[\mathbf{v}]_{\mathcal{B}} = [\mathbf{v}]_{\mathcal{A}}$.

(c) Let $M = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$ be the $\mathcal{B}$-matrix of a linear map $T\colon \mathbb{R}^2 \to \mathbb{R}^2$. What is the $\mathcal{A}$-matrix $N = [T]_{\mathcal{AA}}$ of T?

(a)

$$[1]_{\mathcal{BA}} = \left(\left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} \right]_{\mathcal{B}} \left[\begin{pmatrix} 3 \\ 2 \end{pmatrix} \right]_{\mathcal{B}} \right) = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}.$$

(b)

$$[1]_{\mathcal{AB}} = [1]_{\mathcal{BA}}^{-1} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}^{-1} = \frac{1}{2 \cdot 2 - 3 \cdot 1} \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}.$$

(c)

$$N = [1]_{\mathcal{AB}} M [1]_{\mathcal{BA}} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} -19 & -34 \\ 13 & 23 \end{pmatrix}.$$

4. True or False. You must explain your answer.

(a) Let M be a matrix. If the kernel of M equals $\{\mathbf{0}\}$, then the columns of M are linearly independent.

(b) If V is a subspace of $\mathbb{R}^n$ and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, then $2\mathbf{u} - 3\mathbf{v} + 4\mathbf{w} \in V$ also.

(a) True. Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be the columns of M and let

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

be a vector in $\mathbb{R}^n$. Then

$$M\mathbf{x} = x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n$$

is the linear combination of the columns of M with coefficients the entries x_i of $\mathbf{x}$. So, if $\ker(M) = \{\mathbf{0}\}$, then

$$x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n = \mathbf{0} \Rightarrow x_1 = \cdots = x_n = 0.$$

That is, $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent.

(b) True. A subset $V \subset \mathbb{R}^n$ is a *subspace* if $\mathbf{0} \in V$ and V is closed under addition and scalar multiplication. In particular, any linear combination of elements of V (such as $2\mathbf{u} - 3\mathbf{v} + 4\mathbf{w}$) is also an element of V.

5. Let S be the set of 2×2 matrices A such that

$$A \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Find a basis for S. What is the dimension of S?

Write

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then

$$A \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} a+b & a+b \\ c+d & c+d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

gives $a + b = 0$ and $c + d = 0$. So $b = -a$ and $d = -c$, and we have

$$S = \left\{ \begin{pmatrix} a & -a \\ c & -c \end{pmatrix} \;\middle|\; a, c \in \mathbb{R} \right\} = \operatorname{Span}\left(\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \right).$$

We see that

$$\left\{ \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \right\}$$

is a basis of S (it spans S and is linearly independent). The dimension of a vector space is the size of a basis, so the dimension of S equals 2.

6. Let $\mathcal{P}_2$ denote the vector space of polynomials of degree less than or equal to 2. Find a basis for the image and kernel of the linear map

$$T\colon \mathcal{P}_2 \to \mathbb{R}^2, \quad T(f) = \begin{pmatrix} f(1) \\ f'(1) \end{pmatrix}.$$

Write $f(t) = a_0 + a_1 t + a_2 t^2$. Then $f'(t) = a_1 + 2a_2 t$. So

$$T(f) = \begin{pmatrix} f(1) \\ f'(1) \end{pmatrix} = \begin{pmatrix} a_0 + a_1 + a_2 \\ a_1 + 2a_2 \end{pmatrix}$$

A basis of the kernel is given by solving the system of linear equations

$$\begin{aligned} a_0 + a_1 + a_2 &= 0 \\ a_1 + 2a_2 &= 0 \end{aligned}$$

We find (by Gaussian elimination, or inspection)

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = a_2 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

So the kernel of T has basis $\{1 - 2t + t^2\}$ given by the polynomial corresponding to the vector

$$\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

By the rank-nullity formula, the image of T has dimension 2, so $\operatorname{im}(T) = \mathbb{R}^2$, and we can take the standard basis

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

7. Let $\mathcal{A}$ be the basis $\{1, t, t^2\}$ of $\mathcal{P}_2$, the set of all polynomials of degree less than or equal to 2. Find the $\mathcal{A}$-matrix of the linear map

$$T\colon \mathcal{P}_2 \to \mathcal{P}_2, \quad T(f) = tf' + 2f'' - f.$$

Write $f(t) = a_0 + a_1 t + a_2 t^2$. Then $f'(t) = a_1 + 2a_2 t$ and $f''(t) = 2a_2$, so

$$T(f) = t(a_1+2a_2t)+2\cdot(2a_2)-(a_0+a_1t+a_2t^2) = (4a_2-a_0)+a_2t^2 = a_0(-1)+a_1(0)+a_2(4+t^2).$$

So the $\mathcal{A}$-matrix of T is the matrix

$$[T]_{\mathcal{A}\mathcal{A}} = \begin{pmatrix} -1 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

whose columns are the $\mathcal{A}$-coordinate vectors of the polynomials $T(1), T(t), T(t^2)$ (these are the polynomials $-1, 0, 4 + t^2$ occuring with coefficients a_0, a_1, a_2 in the expression for $T(f)$ above).