Linear Algebra Midterm II Review Sheet

Prepared by your generous peer, Jonah Palmer, April 2017 (my comments are in red – Rob Kusner ;-)

# 3.1:

**Easy 3x3 determinant equation:**

**Computing the determinant using cofactor expansion across rows / columns:**

* Note: You should get the same answer for every row or column you use cofactor expansion on.

Remember the checkerboard of ± signs in the cofactors

Here is the minor matrix resulting from removing the row i and col j from

**Row operations and the effect on the determinant:**

* It multiplied the determinant by 7.

* It does not change the determinant.

* The determinant is multiplied by .

**Finding a formula relating to and :**

* Note: Notice the example at the top of the page. When one row is filled with k’s, the determinant is multiplied by . However, when two rows are multiplied by k’s, then the determinant is .

# 3.2:

**Properties of Determinants:**

1. A multiple of one row of is added to another row to produce matrix , then:
2. If two rows are interchanged to produce a matrix , then:
3. If one row is multiplied by to produce a matrix , then:
4. If are both matricies then:

* Note:

**True / False Statements:**

1. A row replacement operation does not affect the determinant of a matrix. i.e. adding some row to another row
   1. TRUE
2. The determinant of A is the product of the pivots in any echelon form U of A, multiplied by , where r is the number of row interchanges made during row reduction from A to U.
   1. FALSE
      1. Reduction to an echelon form may also include scaling a row by a nonzero constant, which will change the value of the determinant.
3. If the columns of A are linearly dependent, then .
   1. TRUE
      1. If columns of A are linearly dependent, A is not invertible.
   2. FALSE
      1. Not one of the properties!

**Showing equivalent equations using determinants:**

# 3.3:

**Finding Area using determinants:**

Find the area of the parallelogram whose vertices are listed:

* Note: The area of the parallelogram determined by the columns of is .
* Note: We can use any two vertices here; we’ll use .

The area of the parallelogram is 13 square units. Note that the sign of the determinant gives info about the “right” or “left” handedness (orientation) of the parallelotope.

**Finding volume using determinants:**

Find the volume of the parallelepiped with one vertex at the origin and adjacent vertices at .

* Note: If is a 3x3 matrix, the volume of the parallelepiped determined by the columns of is .

The volume of the parallelepiped is 76.

**Computing the area of an image of under the mapping :**

Let S be the parallelogram determined by the vectors and , and let . Compute the area of the image of S under the mapping .

# 4.1:

**Sets of vectors with inclusion and exclusion:**

Let V be the set of vectors shown below.

It helps to draw pictures of these sets to see what’s going on, especially when looking for a counter-example to show a property fails.

*If u and v are in V, is u + v in V? Why?*

, and , then

Since and , then , which implies

Since and , then , which implies

Since , this means that u + v is in V.

*Find a specific vector u in V and a specific scalar c such that cu is not in V.*

* Note: Any vector for u will work, make one up.

, then

* Note: cu will not be in V if . So, find a c where this is true. Ex. .

If , and , then cu will not be in V.

**A subspace of a vector space V is a subset H of V properties:**

1. The zero vector of V is in H. This is equivalent to V being nonempty.
2. H is closed under vector addition. That is, for each u and v in H, the sum u + v is in H.
3. H is closed under scalar multiplication. This is, for each u in H and each scalar c, the vector cu is in H.

**Showing H is not a subspace of :**

Find two vectors and a vector and a scalar to show that H is not a subspace of . Picture! (Ellipse!)

* Note: For two vectors to show that H is not a subspace of , they must show that H is not closed under addition.

is not in H.

* Note: For a vector and a scalar to show that H is not a subspace of , they must show that H is not closed under scalar multiplication.

is in H. The vector cu is equal to

Say, . Find a vector u such that cu is not in H.

* Note:
* Note:

**Spanning and vectors:**

Let H be the set of all vectors of the form . Find a vector v in such that .

* Note: Due to the theorem; The Span of any nonempty set is always a subspace!

*If are in a vector space V, then is a subspace of V.*

We can say because v is in and , H is a subspace of

**Determining subspace of vectors:**

Is w in the subspace spanned by

* Note: To determine if w is in the subspace spanned by , we must determine if w is a linear combination of .
* Note: No row has the form where b is nonzero. Therefore, w is a linear combination of

**Spanning Sets:**

Find a set S of vectors that spans W.

**Vector Space Axioms:**

1. The sum u + v is in V Some consider this to be a “given” rather than an “axiom”….
2. u + v = v + u
3. (u + v) + w = u + (v + w)
4. V has a vector – such that u + 0 = u
5. For each u in V, there is a vector -u in V such that u + (-u) = 0
6. The scalar multiple cu is in V Some consider this to be a “given” rather than an “axiom”….
7. c(u + v) = cu + cv
8. (c + d)u = cu + du
9. c(du) = (cd)u
10. 1u = u

3), 4) and 5) are the axioms of a “group” and including 2) makes it a commutative (abelian) group

7) and 8) are distributive laws relating scalar “+” (one of these) with vector “+” (three of these)

10) keeps things from collapsing (without it, the “trivial” scalar multiplication rule “cv=0” for all scalars c and vectors v would satisfy the other axioms)

# 4.2:

**Null space of an m x n matrix A:**

Determine if is in , where

Also called the “kernel”

* Note: To test if w satisfies computer Aw. If the product Aw is 0, then w is in .

, so w is not in .

**Explicit Description of Nul A by Listing Vectors that span the null space:**

* Note: Need to find the general solution of Ax = 0 in terms of free variables.

Spanning set for Nul A =

**Show a given set is a vector space:**

* Note: this given set represents the set of all solutions to the homogeneous system of equations.
  + Therefore; the set
* Note: The null space of an m x n matrix A is a subspace of . Equivalently, the set of all solutions to a system of m homogeneous linear equations in n unknowns is a subspace of .

**Nonzero Vectors in Nul A and Col A:**

First find the general solution of Ax = 0 in terms of free variables.

* Note: To find a nonzero vector in Nul A, choose a nonzero value for the free variable, say

, and substitute into the general solution.

* Note: To find a nonzero vector in Col A, choose any column of A as the nonzero vector.

**Determine if a vector is in Col(A) / Nul(A):**

and . Determine if w is in Col(A) / Nul(A)?

Also called the “image”

* Note: If the equation Ax = w is consistent, then w is in Col(A).
* Note: Since the echelon form of the augmented matrix has no row of the form the equation is consistent. Thus, w is in Col(A).
* Note: If the equation Aw = 0 is true, then w is in Nul(A).

w is in Nul(A).

**True / False Statements:**

1. A null space is a vector space.
   1. TRUE
2. The column space of an m x n matrix is in
   1. TRUE
3. The column space of A, Col(A), is the set of all solutions of Ax = b.
   1. FALSE
4. The null space of A, Nul(A), is the kernel of the mapping .
   1. TRUE
      1. The kernel of a linear transformation T, from a vector space V to a vector space W, is the set of all u in V such that . Thus the kernel of a matrix transformation is the null space of A.
5. The range of a linear transformation is a vector space. Again, most folks call this the “image”.
   1. TRUE
      1. The range of a linear transformation T, from a vector space V to a vector space W, is a subspace of W.
6. The set of all solutions of a homogeneous linear differential equation is the kernel of a linear transformation.
   1. TRUE
      1. The linear transformation maps each function f to a linear combination of f and at least one of its derivatives, exactly as these appear in the homogeneous linear differential equation.

# 4.3:

**Invertible Matrix Theorem:**

Let A be a square n x n matrix. Then the following statements are equivalent.

1. The matrix A is an invertible matrix.
2. The matrix A is row equivalent to the n x n identity matrix.
3. The matrix A has n pivot positions.
4. The equation Ax = 0 has only the trivial solution.
5. The columns of A form a linearly independent set.
6. The columns of A span

**Determining if a set is a basis:**

Determine whether the set is a basis for .

* Note: The columns of A form a linearly independent set and they span
* Note: The set is a basis if the set is linearly independent and they span

**Basis of a null space of a matrix:**

Find the basis for the null space of the matrix.

* Note: The basis for the null space of the matrix will only be the basis if and are linearly independent. They are independent however, because neither is a multiple of the other.

**True / False Statements:**

1. A linearly independent set in a subspace H is a basis for H.
   1. FALSE
      1. The subspace spanned by the set must also coincide with H. One can extend to a maximal linearly independent set – that extended set will be a basis – this is how one shows any vector space has a basis.
2. If a finite set S of nonzero vectors spans a vector space V, then some subset of S is a basis for V. Dually, a minimal spanning set is a basis, but this works easily only in the finite-dimensional case.
   1. TRUE
      1. Spanning Set Theorem
3. A basis is a linearly independent set that is as large as possible. (see above)
   1. TRUE
4. The standard method for producing a spanning set for Nul A sometimes fails to produce a basis for Nul A.
   1. FALSE
      1. The method always produces an independent set.
5. If B is an echelon form of a matrix A, then the pivot columns of B form a basis for Col A.
   1. FALSE
      1. The columns of an echelon form B of A are not necessarily in the column space of A.

# 4.4:

**Finding vector x using a given basis and a coordinate vector :**

Find vector x determined by the given coordinate vector and the given basis .

**Finding the coordinate vector given vector x and basis :**

Find the coordinate vector of x relative to the given basis .

* Note: We are looking for the vector here.

**Using an inverse matrix to find :**

Use an inverse matrix to find for the given x and .

Recall:

**Finding coordinate vectors relative to a set :**

The set is a basis for . Find the coordinate vector of relative to .

* Note: To find the coordinate vector, split up whole numbers, ’s of the first degree, and ’s of the second degree.

- Whole numbers here!

- ’s of the first degree here!

-’s of the second degree here!

* Note: Now we need to take this system of equations and form an augmented matrix.

**Finding matrix A using the inverse of :**

Let . Since the coordinate mapping determined by B is a linear transformation from into , this mapping must be implemented by some 2x2 matrix A. Find it.